

Lecture 5: Cohomology II

Some homological algebra

Defⁿ Given a seq. of ab. grps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we call it exact if $\ker g = \text{im } f$.

Call

$$\dots \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$$

exact if exact at every spot (i.e. $A_2 \rightarrow A_2 \rightarrow A_3$

$$A_0 \rightarrow A_1 \rightarrow A_2$$

$$A_2 \rightarrow A_3 \rightarrow A_4$$

...

Call this a "long exact sequence" (LES).

A LES of the form

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is called a short exact sequence

Ex

$$0 \rightarrow A \xrightarrow{f} B \rightarrow 0$$

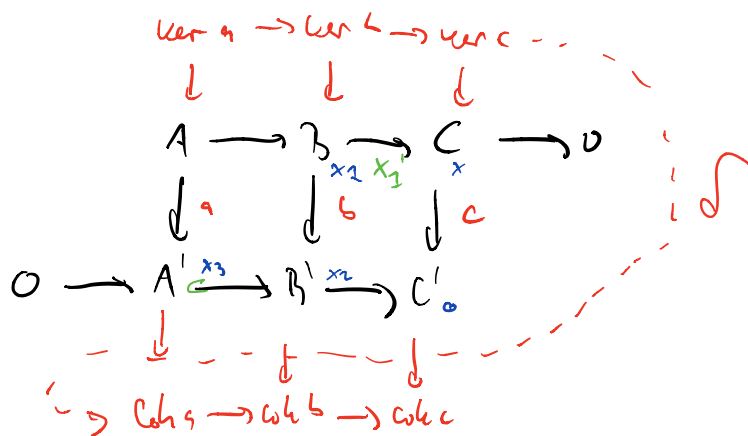
is a "LES" $\Leftrightarrow f$ iso

Ex $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$

- is a SES \Leftrightarrow
- f injective
 - $\ker g = \text{im } f$
 - g surjective
- i.e. $A \hookrightarrow B$
 $C \cong B/A$

Prop² (Snake Lemma)

Suppose given a commutative diagram of 2S. gms



$$\begin{aligned}
 & x_1' - x_2 \in A \\
 & b(x_2) - b(x_1') \in A' \\
 & [b(x_2)] = [b(x_1')] \\
 & \in \text{coker}(a)
 \end{aligned}$$

s.t. rows are exact. There is a construction of a map σ s.t. the induced 6 from sequence of ker to coker is exact.

Proof - watch the movie ?!

Construct σ : Pick $x \in \ker c \subset C$
 Choose some lift $x_1 \in B$
 \cap

Let $x_2 = \delta(x_1) \in B'$
 Then $x_2 \mapsto 0 \in C'$

\therefore may choose $x_3 \in A'$ s.t. $x_3 \mapsto x_2 \in B'$
 Put $\delta(x) = [x_3] \in \text{cok } \alpha$.

Tedious verifications: - δ well-defined
 - seq. is exact at every spot $\frac{\text{spot}}{B}$

Def A cochain complex C' is a seq. of d .
 gms.

$$\dots \rightarrow C^i \xrightarrow{d_i} C^{i+1} \rightarrow \dots$$

s.t. $d_{i+1} \circ d_i = 0 \quad \forall i$.

Then $H^i(C') = \frac{\ker d_i}{\text{im } d_{i-1}}$
 are called cohom. gms. of
 the C' .

A morphism of cochain cx.

$$\text{is } \varphi: C' \rightarrow D'$$

where $\varphi^i: C^i \rightarrow D^i$ is a hom. of ab gms

$$\& \quad C^i \xrightarrow{d_i} C^{i+1}$$

$$\begin{array}{ccc} \varphi^i \downarrow & G & \downarrow \varphi^{i+1} \\ D^i \xrightarrow{d_i} & & D^{i+1} \end{array} \quad \forall i.$$

Ex $C' = C'(G, A)$

G prof. gp

A abscn G -mod

is a cochr. cx.

Ex $\varphi: A \rightarrow B \in \mathcal{C}_G$

$\rightsquigarrow \varphi': C^*(G, A) \rightarrow C^*(G, B)$

is a morph. of cochr. CX.

Propⁿ Let $0 \rightarrow A^i \rightarrow B^i \rightarrow C^i \rightarrow 0$

be an exact seq. of cochr. CX

[i.e. $A^i \rightarrow B^i, B^i \rightarrow C^i$ are morph. of cochr. CX.

$d \circ A^i \rightarrow B^i \rightarrow C^i \rightarrow 0$ exact $\forall i$].

Then obtain LES

$$\begin{array}{c} \dots \\ \hookrightarrow H^q(A) \rightarrow H^q(B) \rightarrow H^q(C) \rightarrow \dots \\ \downarrow \end{array}$$

$$\hookrightarrow H^{q+1}(A) \rightarrow H^{q+1}(B) \rightarrow H^{q+1}(C) \rightarrow \dots$$

$\ker = \ker \left(\frac{A^q}{dA^{q-1}} \rightarrow \frac{A^{q+1}}{dA^q} \right) \stackrel{\cong}{=} \ker \left(\frac{A^q}{dA^{q-1}} \rightarrow \frac{A^{q+1}}{dA^q} \right) / dA^{q-1} \subseteq H^q(A)$

PR

$$\frac{A^q}{dA^{q-1}} \rightarrow \frac{B^q}{dB^{q-1}} \rightarrow \frac{C^q}{dC^{q-1}} \rightarrow 0$$

$$\begin{array}{ccc} \downarrow d & \downarrow & \downarrow \\ \ker(A^q \rightarrow A^{q+1}) & \rightarrow & \ker(B^q \rightarrow B^{q+1}) \rightarrow \ker(C^q \rightarrow C^{q+1}) \end{array}$$

$$0 \rightarrow \ker(A^q \rightarrow A^{q+1}) \rightarrow \ker(B^q \rightarrow B^{q+1}) \rightarrow \ker(C^q \rightarrow C^{q+1})$$

- Check that rows are exact.

- Apply Snake lemma (+ isomth theorem) \square

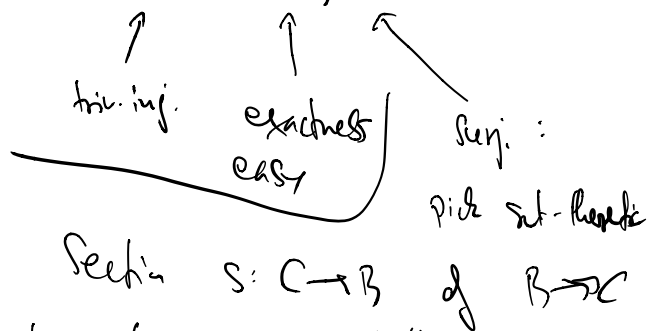
Propⁿ G a prof. gp.

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact seq. of discrete G -mod (i.e. exact as seq. of ab. grps).

Then this LES

$$\begin{array}{c}
 0 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow H^0(C) \rightarrow \\
 \rightarrow H^1(A) \rightarrow H^1(B) \rightarrow H^1(C) \rightarrow \\
 \rightarrow H^2(A) \rightarrow \dots
 \end{array}$$

PF Claim that $0 \rightarrow C^*(G, A) \rightarrow C^*(G, B) \xrightarrow{\alpha} C^*(G, C) \rightarrow 0$ is exact.



$S(c)$ s is cb (D, C discrete!)

$$s_a : C^*(G, C) \rightarrow C^*(B, G)$$

is a section of α . \square

$\mathbb{R} \rightarrow \mathbb{Z}$

$$\begin{array}{c} \xrightarrow{\quad} H^2(A) \rightarrow H^2(K_2 A) \rightarrow \dots \\ \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ \quad \quad \quad \mathbb{Z} \quad \quad \quad \mathbb{Z} \end{array}$$

Th^m (dimension shifting)

Let G, H be profinite groups,

$F: \mathcal{C}_H \rightarrow \mathcal{C}_G$ an exact functor
(i.e. pres. short exact seq.)

preserving induced modules.

Let $\lambda_m: H^m(G, F-) \Rightarrow H^m(H, -)$

[i.e.: for every $A \in \mathcal{C}_H$ provide $\lambda_A: H^m(G, FA) \rightarrow H^m(H, A)$

$$\begin{array}{ccc} \text{s.t. } \forall \varphi: A \rightarrow B \in \mathcal{C}_H, & H^m(G, FA) & \xrightarrow{\rho} & H^m(G, FB) \\ & \downarrow \lambda_A & \circlearrowleft & \downarrow \lambda_B \\ & H^m(H, A) & \xrightarrow{\varphi} & H^m(H, B) \end{array}$$

Then: 1) $\exists!$ seq. of nat. trans.

$$\lambda_i: H^i(G, F-) \Rightarrow H^i(H, -)$$

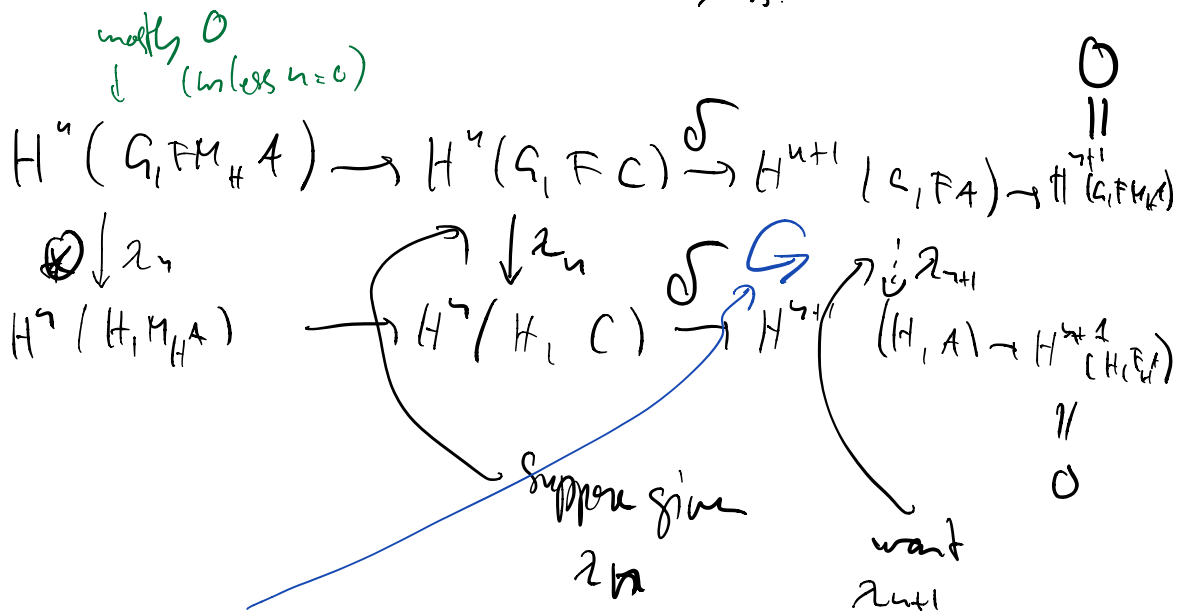
$\forall i \geq m$

commuting with δ .

2) If λ_m is a nat. iso.

then also λ_i is bijective.

$\text{PE } A \in \mathcal{E}_H, \quad 0 \rightarrow A \rightarrow M_H A \rightarrow C \rightarrow 0 \text{ exact}$
 $\rightsquigarrow 0 \rightarrow FA \rightarrow FM_H A \rightarrow FC \rightarrow 0 \text{ exact by ass.}$
 $\underbrace{M_H A}_{M_H A'} \text{ by ass.}$



Commutativity is defⁿ of " λ_i commutes with boundaries"

Surjectivity $\rightsquigarrow \lambda_{n+1}$ is unique if λ_n exists

* : λ_{n+1} exists in this specific diagram

Check: λ_{n+1} is a nat. transf.

λ_{n+1} is an iso if λ_n is

□

Shapiro's theorem Let $H \subset G$ be a closed subgroup. Then for $A \in \mathcal{E}_4$

$$H^n(H, A) \cong H^n(G, M_G^H A).$$

Ex If $H = \{e\}$ then $H^n(H, A) = 0$ for $n > 0$
 & $M_G^{\{e\}} A$ is induced & so $H^n(G, M_G^{\{e\}} A) = 0$.

PE - $M_G^H : \mathcal{E}_H \rightarrow \mathcal{E}_G$ exact functor

- $M_G^H M_H^{\{e\}} A \cong M_G^{\{e\}} A$ i.e. M_G^H pres. induced mod

- $(M_G^H A)^G = \left\{ f: G \xrightarrow{\text{cb}} A \mid \begin{array}{l} f(x) = f(hx) \quad \forall h \in H, x \in G \\ f(gx) = f(x) \quad \forall g \in G \end{array} \right\}$
 $\cong A^H$ i.e. f a constant function at H -invariant elt. of A

Apply dir shifting theorem with $F = M_G^H$ & $M = 0$, \square

Coinstriction Let $H \subset G$ be an open subgroup. Shall construct for $A \in \mathcal{E}_G$

$$\text{cor}: H^n(H, A) \longrightarrow H^n(G, A)$$

$$(\cong H^1(H, \text{Inf}_H^g A))$$

Apply the class shifting theorem to

$$- F: \text{Inf}_H^g: \mathcal{L}_G \rightarrow \mathcal{L}_H$$

$$- \text{Inf}_H^g M, A = \{f: G \rightarrow A\}$$

$$G = \coprod_{\mathcal{C} \in \mathcal{C}^g/H} xH$$

$$\cong \prod_{\mathcal{C} \in \mathcal{C}^g/H} \{f: xH \rightarrow A \text{ cb}\}$$

$$\cong M_H \prod_{\mathcal{C} \in \mathcal{C}^g/H} A \text{ is induced}$$

$$- \alpha_0: A^H \rightarrow A^G$$

$$a \longmapsto \sum_{\mathcal{C} \in \mathcal{C}^g/H} x \cdot a$$

UD: \cdot sum is finite

$$\cdot \mathcal{C} \cap \mathcal{C}' = \emptyset \Rightarrow x = y \cdot h \Rightarrow x \cdot a = y \cdot h \cdot a = y \cdot a$$

\cdot Let $x_1, \dots, x_n \in G$ be good reps, $g \in G$.

$$\text{Have } g \cdot x_i = x_{g(i)} \cdot h_i$$

$$\text{Then } g \cdot \sum_i x_i \cdot a = \sum_i x_{g(i)} \cdot h_i \cdot a$$

2. 4. 1.