Lecture 5: Cohomology II

Some homological algebra

<u>Def</u>

Def^n Given a diag. of ab. grps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we call it <u>exact</u> if $\ker g = \operatorname{im} f$.

Call

$$\cdots \to A_1 \to A_2 \to A_3 \to \cdots$$

exact if exact at every spot (i.e. $A_1 \to A_2 \to A_3$
$A_0 \to A_1 \to A_2$
$A_2 \to A_3 \to A_4$
$\cdots$)

Call this a "long exact sequence" (LES).

A LES of the form

$$0 \to A \to B \to C \to 0$$

is called a short exact sequence

<u>Ex</u>

$$0 \to A \xrightarrow{f} B \to 0$$

is a "LES" $\iff$ f iso

Ex $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

is a SES $\Leftrightarrow$
- f injective   i.e. $A \hookrightarrow B$
- $\ker g = \operatorname{im} f$ $C \cong B/A$
- g surjective

Prop^n (Snake lemma)

Suppose given a commutative diagram of ab. grps

$$
\begin{tikzcd}
& \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[ddll, dashed, "\delta"] & \\
& A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & \\
& \operatorname{coh} a \arrow[r] & \operatorname{coh} b \arrow[r] & \operatorname{coh} c &
\end{tikzcd}
$$

$x_1 \quad x_1' \quad x \quad x_3 \quad x_2$

$x_1' - x_1 \in A$

$b(x_1) - b(x_1') \in A'$

$[b(x_1)] = [b(x_1')] \in \operatorname{coh}(a)$

s.t. rows are exact. There is a construction of a map δ s.t. the induced 6 term sequence of ker & coh is exact.

Proof - watch the movie ?!?

Construct δ: Pick $x \in \ker c \subset C$

Choose some lift $x_1 \in B$

Set $x_2 = \delta(x_1) \in B'$

Then $x_2 \mapsto 0 \in C'$

$\therefore$ may choose $x_3 \in A'$ s.t. $x_3 \mapsto x_2 \in B'$

Put $\delta(x) = [x_3] \in \operatorname{coker} \alpha$.

Tedious verifications: - δ well-defined

- Seq. is exact at every spot.

Def^n A cochain complex $C^\cdot$ is a seq. of ab. grps.

$$\cdots \to C^n \xrightarrow{d_n} C^{n+1} \to \cdots$$

s.t. $d_{i+1} \circ d_i = 0 \quad \forall i$.

Then $H^i(C^\cdot) = \dfrac{\ker d_i}{\operatorname{im} d_{i-1}}$ are called coh. grps. of the $C^\cdot$.

A morphism of cochain cx. is $\varphi^\cdot : C^\cdot \longrightarrow D^\cdot$ where $\varphi^i : C^i \longrightarrow D^i$ is a hom^m of ab. grps

&
$$\begin{tikzcd} C^i \arrow[r, "d_i"] \arrow[d, "\varphi^i"'] \arrow[dr, phantom, "\circlearrowleft"] & C^{i+1} \arrow[d, "\varphi^{i+1}"] \\ D^i \arrow[r, "d_i"] & D^{i+1} \end{tikzcd} \quad \forall i.$$

Ex $C^\cdot = C^\cdot(G, A)$ $\quad$ G prof. grp, A discrete G-mod

is a coch. cx.

$\underline{\text{Ex}}$ $\varphi: A \to B \in \mathcal{C}_G$

$\rightsquigarrow \varphi^\cdot : C^\cdot(G, A) \to C^\cdot(G, B)$

is a mor. of coch. cx.

$\underline{\text{Prop}^n}$ Let $0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$

be an exact sq. of coch. cx

[i.e. $A^\cdot \to B^\cdot$, $B^\cdot \to C^\cdot$ are mor. of coch. cx.

& $0 \to A^i \to B^i \to C^i \to 0$ exact $\forall i$].

Then obtain LES

$$\begin{tikzcd} \cdots \arrow[r] & H^n(A) \arrow[r] & H^n(B) \arrow[r] & H^n(C) \arrow[dll, "\delta"] \\ & H^{n+1}(A) \arrow[r] & H^{n+1}(B) \arrow[r] & H^{n+1}(C) \arrow[r] & \cdots \end{tikzcd}$$

ker $= \ker\left(\frac{A^n}{dA^{n-1}} \to A^{n+1}\right)$

$= \ker(A^n \to A^{n+1}) / dA^{n-1} \cong H^n(A)$

$\underline{\text{Pf}}$

$$\begin{tikzcd} & A^n/dA^{n-1} \arrow[r] \arrow[d, "d"] & B^n/dB^{n-1} \arrow[r] \arrow[d] & C^n/dC^{n-1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(A^{n+1} \to A^{n+2}) \arrow[r] & \ker(B^{n+1} \to B^{n+2}) \arrow[r] & \ker(C^{n+1} \to C^{n+2}) \end{tikzcd}$$

- Check that rows are exact.

- Apply Snake lemma (+ isom^n theorems*) $\square$

<u>Prop^n</u> G a prof. gp.

If $0 \to A \to B \to C \to 0$ is an exact seq. of discrete G-mod (i.e. exact as seq. of ab. gps).

Then obtain LES

$$
\begin{tikzcd}
0 \arrow[r] & H^0(A) \arrow[r] & H^0(B) \arrow[r] & H^0(C) \arrow[dll, out=0, in=180] \\
& H^1(A) \arrow[r] & H^1(B) \arrow[r] & H^1(C) \arrow[dll, out=0, in=180] \\
& H^2(A) \arrow[r] & \cdots &
\end{tikzcd}
$$

<u>Pf</u> Claim that $0 \to C^\cdot(G,A) \to C^\cdot(G,B) \xrightarrow{\alpha} C^\cdot(G,C) \to 0$ is exact.

triv. inj.

exactness easy

surj.: pick set-theoretic section $s: C \to B$ of $B \twoheadrightarrow C$

S/c s is cts (B, C discrete!)

$s\alpha: C^\cdot(G,C) \to C^\cdot(B,G)$

is a section of α. $\square$

Ex $G = C_2$ $\quad K \subset \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{+} \mathbb{Z} \to 0$

$0 \to \mathbb{Z}$ ($K \cong \mathbb{Z}$), $\mathbb{Z}\oplus\mathbb{Z}$ with $\circlearrowleft$ diag., $\mathbb{Z}$ with triv. action

take $(\ )^{G}$: $\quad \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ not surj!

$K^{G} = 0$

$$\therefore\ 0 \to K^{C_2} = 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to H^1(G, K) \to H^1(G, \mathbb{Z}\oplus\mathbb{Z}) \to \dots$$

$\therefore H^1(G,K) = \mathbb{Z}/2$, $\quad H^1(G, \mathbb{Z}\oplus\mathbb{Z}) = 0$ Sle module is induced

NB: maps δ are called boundary maps.

Dimension shifting

Ex $A \in \mathcal{C}_G$. Then $0 \to A \to M_G A \to C \to 0$.

Since $H^n(G, M_G A) = 0$ for $n > 0$ get

$$H^1(G,A) = \frac{H^0(G,C)}{\operatorname{im} H^0(G, M_G A)}$$

$$0 \to H^0(A) \to H^0(M_G A) \to H^0(C) \to H^1(A) \to H^1(M_G A) \to H^1(C) \to$$

$H^1(M_G A) = 0$

$$H^n(G,C) \cong H^{n+1}(G,A)$$

for $n \geq 1$.

$\to H^2(A) \to H^2(M_G A) \to \cdots$
$\quad\quad\quad\quad\quad\quad\quad\ \ \| \; 0$

Th^m (dimension shifting)

Let G, H be profinite groups,

$F : \mathcal{C}_H \longrightarrow \mathcal{C}_G$ an exact functor

(i.e. pres. (short) exact seq.)

preserving induced modules.

Let $\lambda_n : H^n(G, F\,\_\,) \Longrightarrow H^n(H, \_\_)$

i.e.: for every $A \in \mathcal{C}_H$ provide $\lambda_A : H^n(G, FA) \to H^n(H, A)$

s.t. $\forall\ \varphi : A \to B \in \mathcal{C}_H$,

$$\begin{tikzcd}
H^n(G, FA) \arrow[r, "\varphi"] \arrow[d, "\lambda_A"] \arrow[dr, phantom, "\circlearrowleft"] & H^n(G, FB) \arrow[d, "\lambda_B"] \\
H^n(H, A) \arrow[r, "\varphi"] & H^n(H, B)
\end{tikzcd}$$

Then: 1) $\exists!$ seq. of nat. transf.

$\lambda_i : H^i(G, F\,\_\,) \Longrightarrow H^i(H, \_\,)$

$\forall\ i \geq n$

commuting with δ.

2) If λ_n is a nat. iso.

Then also λ_i is isom.

Pf: $A \in \mathcal{C}_H$, $0 \to A \to M_H A \to C \to 0$ exact

$\rightsquigarrow$ $0 \to FA \to \underbrace{F M_H A}_{\overset{\cong}{M_G A'} \text{ by ass.}} \to FC \to 0$ exact by ass.

mostly 0 (unless $n=0$)

$$\begin{tikzcd}
H^n(G, FM_H A) \arrow[r] \arrow[d, "\lambda_n"] & H^n(G, FC) \arrow[r, "\delta"] \arrow[d, "\lambda_n"] & H^{n+1}(G, FA) \arrow[r] \arrow[d, dashed, "\lambda_{n+1}"] & H^{n+1}(G, FM_H A) = 0 \\
H^n(H, M_H A) \arrow[r] & H^n(H, C) \arrow[r, "\delta"] & H^{n+1}(H, A) \arrow[r] & H^{n+1}(H, FM_H A) = 0
\end{tikzcd}$$

suppose given λ_n

want λ_{n+1}

commutativity is def^n of "λ_i commutes with boundaries"

Surjectivity $\rightsquigarrow$ λ_{n+1} is unique if λ_n exists

$*$: λ_{n+1} exists in this specific diagram

Check: - λ_{n+1} is a nat. transf.

- λ_{n+1} is an iso if λ_n is. $\square$

Shapiro's theorem Let $H \subset G$ be a closed subgroup. Then for $A \in \mathcal{C}_H$

$$H^n(H, A) \cong H^n(G, M_G^H A).$$

Ex If $H = \{e\}$ then $H^n(H, A) = 0$ for $n > 0$ & $M_G^{\{e\}} A$ is induced & so $H^n(G, M_G^{\{e\}} A) = 0$.

Pf - $M_G^H : \mathcal{C}_H \to \mathcal{C}_G$ exact functor

- $M_G^H M_H^{\{e\}} A \cong M_G^{\{e\}} A$ i.e. M_G^H pres. induced mod

- $(M_G^H A)^G = \left\{ f : G \xrightarrow{cts} A \;\middle|\; \begin{array}{l} h f(x) = f(hx) \ \forall h \in H, x \in G \\ f(gx) = f(x) \ \forall g, x \in G \end{array} \right\}$

$\cong A^H$ i.e. f a constant function at H-invariant elt. of A

Apply dim shifting theorem with $F = M_G^H$ & $m = 0$. $\square$

Corestriction Let $H \subset G$ be an open subgroup. Shall construct for $A \in \mathcal{C}_G$

$$\mathrm{cor} : H^n(H, A) \longrightarrow H^n(G, A)$$

$(:= H^n(H, \mathrm{Inf}^G_H A))$

Apply the dim shifting thm to

- $F: \mathrm{Inf}^G_H : \mathcal{C}_G \longrightarrow \mathcal{C}_H$

$$G = \coprod_{[x]\in G/H} \overset{\overset{H}{\cong}}{x H}$$

- $\mathrm{Inf}^G_H M_G A = \{ f: G \longrightarrow A \}$

$$\cong \prod_{[x]\in G/H} \{ f: xH \to A \text{ cts} \}$$

$$\cong M_H \prod_{[x]\in G/H} A \quad \text{is induced}$$

- $\lambda_0 : A^H \longrightarrow A^G$

$$a \longmapsto \sum_{[x]\in G/H} x a$$

WD:
- Sum is finite
- $[x]=[y] \Rightarrow x = y\cdot h \Rightarrow xa = yha = ya$
- Let $x_1, \dots, x_n \in G$ be coset reps, $g \in G$.

Have $g \cdot x_i = x_{g(i)} \cdot h_i$

Then $g \cdot \sum_i x_i a = \sum_i x_{g(i)} \cancel{h_i} \cdot a$

$= \sum$