Lecture 4: Cohomology I

Discrete G-modules

<u>Def</u>

Def^n G any group $\cdot$
A any ab. gp $+$

By a G-module structure on A we mean a group hom^n $G \to \mathrm{Aut}_{ab}(A)$

I.e. $\forall g \in G$ $\forall a \in A$ define $g \cdot a \in A$

s.t. $g \cdot (h \cdot a) = (gh) \cdot a$ $\qquad e \cdot a = a$

$g(a+b) = ga + gb$. $\qquad g \cdot 0 = 0$

If G is a top. gp, call A a discrete G-mod if it is a G-module & the action

$$G \times A \to A \quad (g, a) \mapsto g \cdot a$$

is ct. (discrete top.)

I.e. $\forall a \in A$, $\{g \in G \mid ga = a\} \subset G$ is open.

A morphism $\alpha : A \to B$ of G-modules is a hom^n of ab. gps s.t. $\forall g \in G, a \in A$, $\alpha(ga) = g\alpha(a)$.

Denote the resulting category by $\mathcal{C}_G$.

Ex A any ab. gp.

Trivial (discr.) G-module structure $g \cdot a = a$ $\forall g \in G$, $a \in A$

Ex If $A = (\mathbb{Z}/p)^n$. $\mathrm{Aut}(A) = GL_n(\mathbb{F}_p)$

So a G-module structure on A is the same as a cts n-dim^l rep^n of G over $\mathbb{F}_p$.

Inflation & induction

If $\alpha : G \to H$ is a cts hom^m, have a functor $\mathrm{Inf}^H_G : \mathcal{C}_H \longrightarrow \mathcal{C}_G$

$A \longmapsto A$ with action $g \cdot a = \alpha(g) \cdot a$

Often write A instead of $\mathrm{Inf}^H_G A$, when no confusion seems to arise.

Define category $\mathcal{C}$ as follows:

- objects $= \{$pairs $(G, A) \mid G$ prof. gp. A discrete G-mod$\}$
- mor. $(G, A) \longrightarrow (H, B) = \{$pairs $(\alpha, f) \mid$

$\alpha : H \longrightarrow G$ cts.

!! [sic!] $f : \mathrm{Inf}_H^G A \to B \in \mathcal{C}_H \}$

Now let $H \subset G$ be a closed subgroup.

For $A \in \mathcal{C}_H$, define $M_G^H A \in \mathcal{C}_G$

$$\begin{array}{c} \parallel \\ \{ r : G \to A \text{ cts.} \mid r(hx) = hr(x) \ \forall h \in H \} \end{array}$$

Called the 'induced module'.

(obv. G-module structure.)

Important obs: Given $A \in \mathcal{C}_G$

- $A \xrightarrow{u} M_G^H \mathrm{Inf}_H^G A$

  $a \longmapsto r_a$, $r_a(g) = g \cdot a$

  This is injective: $r_a(e) = a$.

  "Any G-module A embeds into an induced module."

- $B \in \mathcal{C}_H$

$$\begin{tikzcd} \mathrm{Hom}_H(\mathrm{Inf}_H^G A, B) \arrow[d, "M_G^H"] \arrow[dr, bend left, "\cong"] & \\ \mathrm{Hom}_G(M_G^H \mathrm{Inf}_H^G A, M_G^H B) \arrow[r, "u^*"] & \mathrm{Hom}_G(A, M_G^H B) \end{tikzcd}$$

check this

"adjunction"

Ex $M^{\{e\}}_G A = \{\text{ch. maps } G \to A\}$ ← any a.s. gp

$ \parallel$

$M_G A$

Def^n of Cohomology

$(G, A) \in \mathcal{E}$

Define a.s. gp $C^n(G, A) = \{\text{ch. maps } G^n \to A\}$
$n \in \mathbb{N}_0$

- pointwise addition
- $C^0(G, A) = A$

Define $d_n : C^n(G, A) \longrightarrow C^{n+1}(G, A)$

$$(d_n f)(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \dots, x_i x_{i+1}, x_{i+2}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n)$$

Exercise $d_{n+1} d_n = 0$

Hence: $\ker(d_{n+1}) \supset \operatorname{im}(d_n)$

$\underline{\text{Def}^n}$ $\quad H^n(G, A) = \dfrac{\ker(d_n)}{\operatorname{im}(d_{n-1})}$

What is this ????

$\underline{\text{Ex}}$ $\quad G = \{e\} \qquad G^n = * \qquad C^n(G, A) = A$

$$\begin{tikzcd}
C^0 \arrow[r] \arrow[d, equal] & C^1 \arrow[r] \arrow[d, equal] & C^2 \arrow[r] \arrow[d, equal] & \cdots \\
A \arrow[r, "0"] & A \arrow[r, "1"] & A \arrow[r, "0"] & A \arrow[r, "1"] & {}
\end{tikzcd}$$

$H^0(\{e\}, A) = A \qquad H^1(\{e\}, A) = 0 \qquad H^2(\{e\}, A) = 0$

$\ldots \quad H^n(\{e\}, A) = 0 \quad \forall n > 0$

Back to general G.

$$\begin{tikzcd}
C^0(G, A) \arrow[r] \arrow[d, equal] & C^1(G, A) \arrow[d, equal] \\
A & \{f : G \xrightarrow{\text{cts}} A\}
\end{tikzcd}$$

$$a \longmapsto d_0(a)(x) = x \cdot a - a$$

$\ker(d_0) = \{a \in A \mid \underbrace{x \cdot a - a = 0}_{x \cdot a = a} \ \forall x \in G\} \subset A$

$=: \underline{A^G}$ "submodule of G-invariants"

$\therefore \boxed{H^0(G, A) = A^G}$

NB: We will see that $H^\bullet(G, A)$ are the "right derived functors" of $A \mapsto A^G$.

Exercise A crossed hom^n $f: G \to A$ is a ch. map s.t. $f(g_1 \cdot g_2) = f(g_1) + g_1 \cdot f(g_2)$. Call f principal if $f(g) = ga - a$, some $a \in A$.

Then $H^1(G, A) = \dfrac{\{\text{crossed hom}^n\text{s}\}}{\{\text{principal crossed hom}^n\text{s}\}}$.

Ex If A is trivial, crossed hom^n = group hom^n

principal $\iff 0$

$\therefore H^1(G, A) = \mathrm{Hom}_{ct}(G, A)$.

NB H^2 related to "group ext^{n}s".

Functoriality Suppose $(\alpha, \varphi) : (G, A) \to (H, B)$

$\in \mathcal{C}$.

Define $(\alpha,\varphi)_*: C^n(G,A) \longrightarrow C^n(H,B)$

$$\left(f: G^n \to A\right) \longmapsto \left(\varphi f: H^n \to B\right)$$

$(\varphi f)(h_1,\dots,h_n)$

$= \varphi f(\alpha h_1,\dots,\alpha h_n)$

$\in B$.

<u>Lemma</u> $(\alpha,\varphi)_*$ is a hom$^{\mathrm{m}}$ of ab. grps &

$$\begin{array}{ccc} C^n(G,A) & \xrightarrow{d_n} & C^{n+1}(G,A) \\ \downarrow{\scriptstyle(\alpha,\varphi)_*} & & \downarrow{\scriptstyle(\alpha,\varphi)_*} \\ C^n(H,B) & \xrightarrow{d_n} & C^{n+1}(H,B) \end{array}$$

commutes.

(proof triv. □)

Hence obtain induced map $H^n(G,A) \xrightarrow{(\alpha,\varphi)_*} H^n(H,B)$.

In other words cohomology $H^n(\text{—})$ is a functor $\mathcal{C} \longrightarrow Ab$.

<u>Ex</u> $\alpha: H \longrightarrow G$

$$\rightsquigarrow \alpha^* = (\alpha, id)_*: H^n(G,A) \longrightarrow H^n(H, \mathrm{Inf}_H^G A)$$

$$\| $$

$$H^n(H,A)$$

"contravariant functoriality in 1st var."

Ex $\varphi : A \to B \in \mathcal{C}_G$

$$\rightsquigarrow \varphi_* = (\mathrm{id}, \varphi)_* : H^n(G, A) \to H^n(G, B)$$

"covariant functoriality in 2nd var."

Prop^n With $G = \lim\limits_{U} G/U$ over open normal subgroups (cofiltered)

$A \in \mathcal{C}_G$.

Consider $A^U \in \mathcal{C}_{G/U}$.

Obtain $(G/U, A^U) \to (G, A) \in \mathcal{C}$,

& hence directed system

$$\{H^n(G/U, A^U)\}_U \longrightarrow H^n(G, A)$$

Then $H^n(G, A) = \operatorname{colim}\limits_U H^n(G/U, A^U)$.

dual notion to lim of lecture 2.

Pf

$$\operatorname{colim}_U \operatorname{Hom}_{ch}((G/U)^n, A^U) \xhookrightarrow{\alpha} \operatorname{Hom}_{ch}(G^n, A)$$

inj. b/c $G^n \twoheadrightarrow (G/U)^n$

$A^U \hookrightarrow A$

just means union of subsets

To prove α an iso:

Given $f: G^n \to A$ cts, $f(G^n) \subset A$
cpct $\therefore$ finite, discrete

$\therefore f(G^n) \subset A^U$ for U suff. small

Continuity of $f \implies$ factors through $(G/U)^n$ for U suff. small.

$\therefore \alpha$ is surj.

Hence: $C^n(G, A) = \operatorname{colim}_U C^n(G/U, A^U)$

Fact: filtered colimit of ab. grps. is exact

Hence result follows from $\circ$. $\square$

Cohomology of induced modules

Recall: $M_G A = M_G^{\{e\}} A$ is called the module induced by A.
G prof. group
A any ab. gp.

Prop^n $H^n(G, M_G A) = 0 \quad \forall n > 0$.

Pf Define $s_n : C^n(G, M_G A) \to C^{n-1}(G, M_G A)$

$$(s_n f)(x_1, \dots, x_{n-1}) : G \to A$$

$$(s_n f)(x_1, \dots, x_{n-1})(x) = f(x, x_1, \dots, x_{n-1})(e).$$

This is a hom^n of ab. gps.

Check: $\boxed{d_{n-1} s_n + s_{n+1} d_n = \mathrm{id}_{C^n}}$ (*) $\forall\, n \geq 1$.

Hence: If $x \in \ker d_n$

$$x = \mathrm{id}_{C^n} x = d_{n-1} s_n x + s_{n+1} \underbrace{\cancel{d_n x}}_{0}$$

$\therefore x \in \mathrm{im}\, d_{n-1}$

Thus $H^n(G, M_G A) = 0$ as needed. $\square$

Rmk (*) is called a chain homotopy.

This structure can often be used to prove $H^* = 0$.

Outlook: Recall that any $A \in \mathcal{C}_G$ has $A \hookrightarrow M_G A$.

look at exact seq. $0 \to A \to M_G A \to C \to 0$.

One might hope to relate $H^*(G, A)$ to $H^*(G, M_G A)$

Thm^n we can hope for this 🙂

Next time: $\hat{H}^n(G, A) \cong \hat{H}^{n-1}(G, C)$