Lecture 12: Galois groups of local fields

Today k is a local field.

$\hat{k}$ = maximal p-ext^n

$G_k = \mathrm{Gal}(\hat{k}/k)$ the pro-p-group of interest

Last time: $H^2(G_k) \hookrightarrow H^2(G_{k(\zeta_p)}) \hookrightarrow \underbrace{(\mathbb{Q}/\mathbb{Z})_p}_{=\mathbb{Z}/p}$ ← p-torsion

I.e. either G_k is free or presented by $\underline{\underline{1}}$ rel^n.

<u>Prop^n</u> $H^1(G_k, \mathbb{Z}/p) \cong (k^\times/p)^\vee$ ← cts dual

<u>Pf</u> $H^1(G_k, \mathbb{Z}/p) = \mathrm{Hom}_{cts}(G_k, \mathbb{Z}/p) = \mathrm{Hom}_{cts}(\mathrm{Gal}(k)_{\cancel{p}}^{\wedge}, \mathbb{Z}/p)$ s/c $\mathbb{Z}/p$ is p-complete

$= \mathrm{Hom}_{cts}(\mathrm{Gal}(k)^{ab}, \mathbb{Z}/p)$

$\underset{\text{C.F.T.}}{\cong} \mathrm{Hom}_{cts}(k^\times, \mathbb{Z}/p) = \mathrm{Hom}_{cts}(k^\times/p, \mathbb{Z}/p)$ $\square$

Need thus understand $k^\times$ (& $k^\times/p$).

Since $\mathcal{O}_k$ is a dvr, every elt of $k^\times$ can be written <u>uniquely</u> as $\pi^n \cdot u$, $n \in \mathbb{Z}$

π fixed uniformizer

$u \in \mathcal{O}_k^\times$.

$\therefore k^\times \simeq \mathbb{Z} \times \mathcal{O}_k^\times$.

$$\mathcal{O}_k^\times \longrightarrow \underbrace{\left(\mathcal{O}_k/m\right)^\times}$$

cyclic group, gen by $\bar{\zeta}$, $\bar{\zeta}^n = 1$ rel^n.

Find $\zeta \in \mathcal{O}_k^\times$ lifting $\bar{\zeta}$. But: lift need not satisfy $\zeta^n = 1$.

By Hensel's lemma, since $\mathcal{O}_k$ is complete & $(n, \operatorname{char} \mathcal{O}_k/m) = 1$, can find such a ζ with $\zeta^n = 1$.

$$\therefore \quad \mathcal{O}_k^\times \cong \left\{\begin{matrix}\text{roots of unity of order coprime} \\ \text{to } \operatorname{char} \mathcal{O}_k/m\end{matrix}\right\} \times E_1$$

$\left\{\ldots\right\} = \langle \zeta \rangle$, $E_1 = \{\ker \mathcal{O}_k^\times \to (\mathcal{O}_k/m)^\times\} = 1 + \pi\mathcal{O}_k$

FROM NOW ON, $\operatorname{Char}(k) = 0$.

Prop^n (Hensel) $\quad E_1 \cong \left(\begin{matrix}\text{roots of unity of order} \\ \text{a power of } p\end{matrix}\right) \times \mathcal{O}_k$

Pf sketch: For simplicity assume that $\widehat{\;\;} = \{1\}$.

Check that $\log : 1 + \pi\mathcal{O}_k \longrightarrow \pi\mathcal{O}_k \cong \mathcal{O}_k$

$$1 + x \longmapsto x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

defines an iso of top. gps. $\square$

$$\therefore \quad k^\times \cong \underset{\pi}{\mathbb{Z}} \times \underset{\text{roots of unity}}{\mu(k)} \times \mathcal{O}_k^\times \quad \text{torsion-free, f.g. } \mathbb{Z}_q\text{-module}$$

$$\therefore \cong \mathbb{Z}_q^{[k:\mathbb{Q}_q]}$$

where $q = \operatorname{char} \mathcal{O}_k/\mathfrak{m}$.

<u>Prop^n</u> If $q \neq p$ then

$$\dim H^2(G_k) = 1 + \delta(k)$$

$$\delta(k) = \begin{cases} 1 & \text{if } \zeta_p \in k \\ 0 & \text{else.} \end{cases}$$

If $q = p$

$$\dim H^2(G_k) = 1 + \delta(k) + [k:\mathbb{Q}_p].$$

<u>Pf</u> $H^2(G_k) = \left(k^\times/p\right)^\vee \sim k^\times/p$

$$\sim \mathbb{Z}/p \times \mu/p \times \left(\mathbb{Z}_q/p\right)^{[k:\mathbb{Q}_q]}$$

$\square$

Digression: Euler characteristics

G a profinite group, A a discrete $\mathbb{F}_p$-G-module.

$$\chi(G, A) = \sum_i (-1)^i \dim H^i(G, A)$$

– defined only if $H^i(G, A) = 0$ for $i \gg 0$
& $\dim H^i(G, A) < \infty$ $\forall i$.

Lemma Let $0 \to A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots A_n \to 0$ be a long exact sequence of $\mathbb{F}_p$-vector spaces.

Then $\sum_i (-1)^i \dim A_i = 0$.

Pf $\ker(\varphi_{i+1}) \cong \operatorname{im}(\varphi_i)$

$$A_i \cong \operatorname{im}(\varphi_i) \oplus \frac{A_i}{\operatorname{im}(\varphi_i)} \cong \operatorname{im}(\varphi_i) \oplus \operatorname{im}(\varphi_{i+1})$$

$$\sum_i (-1)^i \left(\dim \operatorname{im} \varphi_i + \dim \operatorname{im} \varphi_{i+1}\right) = 0$$

b/c all terms cancel & $\varphi_i = 0$ for $i < 0$ & $i > n$. □

Prop^n Given $0 \to A_1 \to A_2 \to A_3 \to 0$ exact seq. of discrete $\mathbb{F}_p$-mod. s.t. $\chi(G, A_i)$ all defined.

Then $\chi(G, A_1) - \chi(G, A_2) + \chi(G, A_3) = 0$

Pf Apply prev. to LES $0 \to H^0(G, A_1) \to H^0(G, A_2) \to H^0(G, A_3) \to H^1(G, A_1) \to \cdots$

& rearrange terms. □

Def^n $\chi_n(G,A) = \sum_{i=0}^{n} (-1)^i \dim H^i(G,A)$.

— only defined if all dimensions are finite.

<u>Lemma</u> Let G be a pro-p-group, A discrete $\mathbb{F}_p$-mod,

$\dim A < \infty$, $\chi_n(G)$ defined.
$\mathbb{Z}/p$

Then $\chi_n(G,A)$ is defined &

$$(-1)^n \chi_n(G,A) \leq (-1)^n \dim(A)\, \chi_n(G).$$

Pf Suppose we have exact seq. $0 \to A_1 \to A \to A_2 \to 0$
& holds for A_1, A_2.

Get LES $0 \to H^0(G,A_1) \to H^0(G,A) \to H^0(G,A_2) \to \cdots$

$\longrightarrow H^n(G,A_2) \longrightarrow C \longrightarrow 0.$

Hence $\dim H^i(G,A) < \infty$ $\forall i \leq n$ & by the lemma

$$\chi_n(G,A_1) - \chi_n(G,A) + \chi(G,A_2) - (-1)^n \cdot \dim C = 0$$

$$\therefore (-1)^n \cdot \chi_n(G,A) = (-1)^n \left(\chi_n(G,A_1) + \chi_n(G,A_2)\right) - \dim C$$

$$\leq (-1)^n \cdot \chi_n(G,A_1) + (-1)^n \cdot \chi_n(G,A_2)$$

$$\leq (-1)^n \cdot \underbrace{(\dim A_1 + \dim A_2)}_{\dim A} \cdot \chi_n(G)\,.$$

by induction

Since G is a pro-p-group, A is a finite extension of $\mathbb{F}_p$-s with trivial action. $\therefore$ reduce to $A = \mathbb{F}_p$ ~ trivial. $\square$

Th^m Let G be a pro-p-group & assume that

$$\chi_n(U) = (G:U) \cdot \chi_n(G)$$

open subgp.

for all U in some system of open nbd of 1.

Then $cd_p G \leq n$.

Pf Seek to show that $H^{n+1}(G) = 0$.

Let $\bar{a} \in H^{n+1}(G)$ be represented by $a \in K^{n+1}(G)$.

See that a is inflated from G/U for some open normal U. $\therefore$ $\bar{a}|_U = 0$.

Consider $0 \to \mathbb{Z}/p \xrightarrow{u} M_G^U \mathbb{Z}/p \to A \to 0$.

Since $H^*(G, M_G^U \mathbb{Z}/p) = H^*(U, \mathbb{Z}/p)$,

$$\underbrace{H^{n+1}(U)}\ (\bar{a}) = 0.$$

$$\varphi : H^{n+1}(G, \mathbb{Z}/p) \to H^{n+1}(G, M_G^U \mathbb{Z}/p)$$

LES: $0 \to H^0(G) \to \cdots \to H^n(G, A) \to \ker(\varphi) \to 0.$

$$\therefore \dim \ker \varphi = (-1)^n \Big[\chi_n(G) + \chi_n(G, A) - \underbrace{\chi_n(G, M_G^U \mathbb{Z}/p)}_{\chi_n(U)}\Big]$$

$$\leq (-1)^n \Big[\chi_n(G) + (\dim A) \cdot \chi_n(G) - \underbrace{\chi_n(U)}_{= (G:U) \cdot \chi_n(G)}\Big]$$

$$\dim A + 1 = (G:U) = \dim M_G^U \mathbb{Z}/p$$

$$= 0.$$

$\therefore \bar{a} = 0$ $\square$

Back to local fields

Thm (1) If $\delta(k)=1$ then $\mathrm{cd}\, G_k = 2$.
(2) If $\delta(k)=0$ then $\mathrm{cd}\, G_k = 1$.

Pf $H^0(G_k, \mathbb{Z}/p) = \mathbb{Z}/p$

$$\dim H^2(G_k, \mathbb{Z}/p) = 1 + \delta(k) + \underbrace{\varepsilon(k)}_{0 \text{ or } [k:\mathbb{Q}_p]}$$

if $\delta(k)=1$ then $H^2(G_k, \mathbb{Z}/p) = \mathbb{Z}/p$.

(1) $\chi_2(G_k) =: \chi_2(k) = -\varepsilon(k)$

$U \subset G_k$ open $\longleftrightarrow$ p-extn k'/k

$\chi_2(U) = \chi_2(k') = -\varepsilon(k')$

Observe that $\varepsilon(k') = [k':k]\cdot \varepsilon(k)$ in all cases.

$\therefore$ theorem applies

(2) $\chi_2(k) = -\varepsilon(k)$

If k'/k is a p-extn then $\delta(k')=0$
b/c $[k(\zeta_p):k] \mid p-1$.

$\therefore$ theorem applies again. $\square$

I.e. if $\delta(k)=0$ then G_k is free of a known rank.

Ex $\delta(k)=0$ $\quad \mathrm{char}(\mathcal{O}_k/\mathfrak{m}) = q \neq p$.

Then $G_k \cong \mathbb{Z}_p$.

Let $k' = k(\zeta_{p^n} \mid n \geq 1)$.

Then $\mathrm{Gal}(k'/k) \hookrightarrow \mathbb{Z}_p^\times$

$$\cong \mathbb{Z}/r \times \mathbb{Z}_p$$

$$r \mid p-1$$

Let $k'' = (k')^{\mathbb{Z}/r}$. Then $\mathrm{Gal}(k''/k) = \mathbb{Z}_p$.

Claim that $k'' = \hat{k}$.

Indeed $k'' \subset \hat{k}$ & hence

$$\begin{array}{ccc} G_k & \longrightarrow & \mathrm{Gal}(k''/k) \\ \| & & \| \\ \mathbb{Z}_p & & \mathbb{Z}_p \end{array}$$

Any such surj^n is iso. ($\therefore$ claim)

Ex $\delta(k)=1$, $q \neq p$.

Then $\dim H^1(G_k) = 1+\delta(k) = 2$

$\dim H^2(G_k) = 1$.

Let k'/k be any finite p-ext^n.

Write as $\underbrace{k'/\underbrace{k'^{ur}/k}_{\text{adj. some roots of units}}}_{\text{purely ramified of deg. } p^n}$

Pick a uniformizer $\hat{\pi}$ of k'.

$\rightsquigarrow (\hat{\pi})^{p^n} = \pi \cdot \xi \cdot a$

$a \in E_1 \rightsquigarrow p$-divisible as we have seen

$\rightsquigarrow \pi \cdot \xi$ is a p^n-th root.

$\therefore \; k' = k'^{ur}\left(\sqrt[p^n]{\pi \cdot \xi}\right)$

One concludes that $\hat{k} = k\left(\xi_{p^n}, \sqrt[p^n]{\pi} \;\middle|\; n \geq 1\right)$.

Standard Galois theory arguments $\rightsquigarrow G_k$ is an infinite dihedral group, i.e. generated by σ, τ, rel^n $\boxed{\sigma\tau\sigma^{-1} = \tau^{|\mathcal{O}_k/\mathfrak{m}|}}$

NB: only remaining case char $\mathcal{O}_k/\mathfrak{m} = p$, $\delta(k) = 1$

— not completely known.