Lecture 11: maximal p-extensions

Fix a prime p.

Def^n k any field. $\hat{k} \subset k^s$ is the union of all (finite) Galois ext^{ns} of degree a power of p.

NB: - $\mathrm{Gal}(k_1 k_2/k) \hookrightarrow \mathrm{Gal}(k_1/k) \times \mathrm{Gal}(k_2/k)$
$\therefore$ colimit is filtered & $\hat{k}$ is a field (infinite Galois ext^n of k)

- Let k'/k be any fin. ext^n of degree a power of p, $K/k'/k$ be a Galois closure of k'/k.
$\mathrm{Gal}(K/k') \hookrightarrow \mathrm{Gal}(K/k)$
has index a power of p (namely $[k':k]$).
$\therefore \exists N \subset \mathrm{Gal}(K/k')$ normal in $\mathrm{Gal}(K/k)$, idx a power of p.
$\therefore K^N \supset k' \supset k$ is a Galois p-ext^n.

<u>Hence</u> $\hat{k}$ is closed under separable p-ext^{ns}.

Put $G_k = \mathrm{Gal}(\hat{k}/k)$. This is a pro-$p$-group.

Goal: - find cases where G_k is free,
i.e. $H^2(G_k) = H^2(G_k, \mathbb{Z}/p) = 0$.

- determine the # of gens, i.e.

$\dim H^1(G_k)$.

Fields of char. p

Thm Let $\mathrm{char}(k) = p$. Then G_k is free of rank $\dim k^+/\alpha(k)$, $\alpha : k \to k$, $x \mapsto x^p - x$.

Pf Artin–Schreier sequence $0 \to \underset{\mathbb{F}_p}{\overset{\shortparallel}{\mathbb{Z}/p}} \to \hat{k} \xrightarrow{\alpha} \hat{k} \to 0$.

This is exact: – $\ker(\alpha) \subset \hat{k}$ set of sol^{ns} of $x^p - x = 0$.
$\therefore$ at most p sol$^{(ns)}$
$\therefore \ker = \mathbb{F}_p$

– surjectivity of α: Suppose $t \in \hat{k}$. Need to solve $\underbrace{X^p - X - t}_{\text{separable}} = 0$.
$\therefore$ sol^{ns} generate a sep. p-ext^n of $\hat{k}$.
But $\hat{k}$ closed under such ext^{ns}, $\therefore X \in \hat{k}$.

$\rightsquigarrow$ LES

$$\begin{array}{ccccc} H^0(G_k, \mathbb{F}_p) & \to & H^0(G_k, \hat{k}) & \xrightarrow{\alpha} & H^0(G_k, \hat{k}) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{F}_p & & k & x \mapsto x^p - x & k \end{array}$$

$$\hookrightarrow H^1(G_k, \mathbb{F}_p) \to \cancel{H^1(G_k, \hat{k})} \xrightarrow{\alpha} \cancel{H^1(G_k, \hat{k})} \to$$

$$\hookrightarrow H^2(G_k, \mathbb{F}_p) \longrightarrow H^2(G_k, \bar{k}^2) \longrightarrow \cdots$$
$$\overset{\shortparallel}{0} \qquad\qquad 0 \qquad 0 \qquad 0$$

This concludes. $\square$

From now on: $\operatorname{char}(k) \neq p$.

Fields containing ζ_p primitive p^{th} root of unity

Thm $H^2(G_k, \mathbb{Z}/p) \cong H^2(G_k, \bar{k}^\times)_p$ p-torsion

$\dim H^2(G_k, \mathbb{Z}/p) = \dim k^\times/p$

Pf Kummer sequence $0 \longrightarrow \mathbb{Z}/p \xrightarrow{\varphi} \bar{k}^\times \xrightarrow{p} \bar{k}^\times \longrightarrow 1$.

$\varphi(n) := \zeta_p^n$

As before this is exact: - ker (p) = {p-th roots of unity in $\bar{k}^\times$}
$= \mathbb{Z}/p$ by ass.

- surjectivity of p: $X^p - t = 0$ is a sep. eqn of degree p.

$$\text{LES} \rightarrow \underset{\substack{\shortparallel \\ k^\times}}{H^0(G_k, \bar{k}^\times)} \longrightarrow \underset{\substack{\shortparallel \\ k^\times}}{H^0(G_k, \bar{k}^\times)} \rightarrow \underset{\substack{\shortparallel \\ \therefore\ k^\times/p}}{H^1(G_k, \mathbb{Z}/p)} \rightarrow \underset{\substack{\shortparallel \\ 0}}{H^1(G_k, \bar{k}^\times)}$$

$$\underset{\shortparallel}{H^1(G_k, \bar{k}^\times)} \longrightarrow \underset{\shortparallel}{H^2(G_k, \mathbb{Z}/p)} \rightarrow H^2(G_k, \bar{k}^\times) \xrightarrow{p} H^2(G_k, \bar{k}^\times)$$

$0 \qquad \therefore \overset{\shortparallel}{H^2(G_k, \hat{k}^\times)_p}$. $\square$

Def^n Call a local field k <u>p-infinite</u> if $\forall n \geq 0$ there exists local field $k' \subset k$ s.t. $p^n \mid [k : k']$.

I.e. k is "sufficiently big".

Th^m If k is a p-infinite local field ($\&$ $\mathrm{char}(k) \neq p$ $\&$ $\zeta_p \in k$) then G_k is free.

Pf TS $H^2(G_k, \mathbb{Z}/p) = 0$
$$\cong H^2(G_k, \hat{k}^\times)_p$$

Write $k = \mathrm{colim}_i\, k_i$, where $\{k_i\}$ is a filtered family of local subfields s.t. $\forall i\ \exists j \geq i$ s.t. $p \mid [k_j : k_i]$.

Then $G_k \cong \lim_i G_{k_i}$ (i.e. $\hat{k} = \mathrm{colim}\, \hat{k}_i$)

$\&$ $H^2(G_k, \hat{k}^\times)_p = \mathrm{colim}_i\, H^2(G_{k_i}, \hat{k}_i^\times)_p$.

Class field theory:

$$\begin{tikzcd}
H^2(G_{k_i}, \hat{k}_i^\times)_p \arrow[r, hook, "\mathrm{inv}"] \arrow[d, "\therefore 0"'] \arrow[dd, bend right, "\therefore 0"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot [k_j : k_i] \leftarrow \text{div. by } p"] \\
H^2(\mathrm{Gal}(\hat{k}_i k_j / k_j), (\hat{k}_i k_j)^\times)_p \arrow[r, hook, "\mathrm{inv}"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \\
{} &
\end{tikzcd}$$

$H^2(G_{k_i}, \hat{k}_i)_p$. The result follows. $\square$

Thm Let k be a global field (char $\neq p$ ...) such that k_v is p-infinite for every finite place v.
If $p=2$, assume k totally imaginary (i.e. all infinite places complex). Then G_k is free.

Pf Hasse principle: $H^2(G_k, \hat{k}^\times)_p \hookrightarrow \bigoplus_v H^2(G_{k_v}, \hat{k}_v^\times)_p$

finite & infinite places

Want LHS $= 0$
suffices RHS $= 0$.
If v is finite, $H^2(G_{k_v}, \hat{k}_v^\times)_p = 0$ by prev. thm.
If v infinite: $p=2$ then $k_v = \mathbb{C}$, $G_{k_v} = \{1\}$, $H^2 = 0$
$p \neq 2$ then $\mathrm{Gal}(k_v) = \{1\}$ or $\mathbb{Z}/2$.
$\therefore H^2(G_{k_v}, \hat{k}_v^\times)$ is 2-torsion
$\therefore H^2(G_{k_v}, \hat{k}_v^\times)_p = 0$. $\square$

Example Suppose k is local or global field containing ζ_{p^n} $\forall n$. (E.g. $\mathbb{Q}(\zeta_{p^n} \mid n)$.)

Then G_k is free.

Fields not ctg ζ_p

Put $k' = k(\zeta_p)$.

NB: $[k':k] \mid p-1$, in particular coprime to p.

$\mathrm{Gal}(k'/k) \hookrightarrow \mathbb{F}_p^\times$

Thm Suppose that $H^2(\mathrm{Gal}(\hat{k'}/k)) \to H^2(\mathrm{Gal}(\hat{k'}/\hat{k}k'))$ is the zero map.

Then $H^2(G_k) \cong H^2(G_{k'})^{\mathrm{Ga}(k'/k)}$.

Pf Step 1: $H^2(\mathrm{Gal}(\hat{k}'/\hat{k}), \mathbb{Z}/p) = 0$.

Let $\lambda \in \mathrm{Gal}(\hat{k}'/\hat{k}') \longrightarrow \mathbb{Z}/p$.

Then $\ker \lambda$ has index 1 or p.

If idx $= p$, then $\ker \hookrightarrow L/\hat{k}$ of degree p.

does not exist.

$\therefore \ker\lambda = \mathrm{Gal}$ $\quad \therefore \lambda = 0$.

Step 2 Higher transgression sequence:

If $H \subset G$ is a closed normal subgp. of a profinite gp,

$A \in \mathcal{C}_G$, $H^i(H,A)=0$ for $0<i<n$.

Then $\exists$ exact sequence

$$0 \to H^n(G/H, A^H) \xrightarrow{\text{inf}} H^n(G,A) \xrightarrow{\text{res}} H^n(H,A)^{G/H} \xrightarrow{\text{tra}} H^{n+1}(G/H, A^H) \xrightarrow{\text{inf}} H^{n+1}(G,A)$$

(Pf similar to case $n=1$ from before.)

Cor: If $|G:H|$ is finite & all elts of A have order coprime to $|G:H|$, then

$$H^n(G/H, A^H) = 0 \ \forall n>0 \ \& \ H^n(G,A) \cong H^n(H,A)^{G/H}.$$

Pf 1st statement $\Rightarrow$ 2nd statement by transgression seq. mult^n by $|G:H|$ is 0 on $H^n(G/H, A^H)$ but also iso b/c $A^H \subset A$ is of torsion prime to $|G:H|$.

$\therefore H^n(G/H, A^H) = 0$ $\square$

Conclusion of proof:

Since $[\hat{k}k' : \hat{k}]$ is prime to p ($\hat{k}$ has no p-ext^{ns}!) we have

$$H^2(\mathrm{Gal}(\hat{k}'/\hat{k})) \hookrightarrow H^2(\mathrm{Gal}(\hat{k}'/\hat{k}k')).$$

Hence

$$H^2(\mathrm{Gal}(\hat{k}/k)) \xrightarrow{\therefore 0} H^2(\mathrm{Gal}(\hat{k}'/\hat{k})) \hookrightarrow \ldots$$

$$\xrightarrow[\text{0 by ass.}]{} H^2(\mathrm{Gal}(\hat{k'}/\hat{k}k')).$$

Higher transgression sequence:

$$0 \to H^2(\mathrm{Gal}(\hat{k}/k)) \xrightarrow[\therefore\ \cong]{\text{inf}} H^2(\mathrm{Gal}(\hat{k'}/k)) \xrightarrow[0]{\text{res}} H^2(\mathrm{Gal}(\hat{k'}/\hat{k}k'))$$

$$\begin{array}{c} H^2(\mathrm{Gal}(\hat{k'}/k)) \\ \| \text{S by cor.} \\ H^2(\mathrm{Gal}(\hat{k'}/k'))^{\mathrm{Gal}(k'/k)} \end{array}$$

$\square$

<u>Thm</u> Let k be a local field which is p-infinite, or k a global field s.t. all completions at finite places are p-infinite & $p \neq 2$.
Then G_k is free.

<u>Pf</u> k' & $\hat{k}k'$ contain ζ_p & satisfy other assumptions.

$\therefore$ $H^2(\mathrm{Gal}(\hat{k'}/k'\hat{k})) = 0$ $\quad\therefore$ we may apply prev. thm

$H^2(G_{k'}) = 0$ $\quad$ i.e. $H^2(G_k) = H^2(G_{k'})^{\mathrm{Gal}(k'/k)}$

$= 0$ $\square$