Lecture 10: Class field theory

Frobenius

Thm K/k Galois, v a finite place of K.

$$D_v \longrightarrow \mathrm{Gal}(K(v)/k(v))$$

residue fields

is a surjection with kernel I_v.

Pf omitted.

Now if $k(v)$ is finite, then $\hat{\mathbb{Z}} \twoheadrightarrow \mathrm{Gal}(K(v)/k(v))$

$$1 \longmapsto \text{Frobenius}$$
$$x \longmapsto x^{|k(v)|}$$

If v is unramified (i.e. $I_v = \{e\}$) then

$$\begin{array}{ccc} D_v & \cong & \mathrm{Gal}(K(v)/k(v)) \\ \psi & & \psi \\ \sigma_v & & \text{Frobenius} \end{array}$$

"Frobenius element"

Lemma If $\overset{\text{Galois}}{K/k}$ is unramified at v, & $K/\underbrace{L/k}_{\text{Galois}}$,

$w = v|_L$. Then $\sigma_v|_L = \sigma_w$.

Pf clear. $\square$

Local fields & global fields

A *global field* is an algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_p(x)$. It is called finite global field if the ext^n is finite. A local field is an algebraic ext^n of $\mathbb{Q}_p$ $(= \mathrm{Frac}(\hat{\mathbb{Z}}_p))$ or $\mathbb{F}_p((x))$. Finite local field if ext^n is finite.

NB: K a global field, v a finite place, then K_v is a local field.
$\therefore$ often discuss $\mathbb{R}, \mathbb{C}$ along with local fields

In this lecture: all fields local or global.

Class groups

Let k be a finite global field.

Def^n $\mathcal{D}_k$ = group of divisors in $\mathcal{O}_k$
= free abelian group on finite places

Have a hom^m $k^\times \longrightarrow \mathcal{D}_k$

$$x \longmapsto \sum_v \underbrace{v(x)}_{|} v$$

$\mathcal{O}_{k,v}$ is a dvr. Pick uni formizer π.

$x = u \cdot \pi^{v(x)}$

The cokernel is called the (ideal) <u>class group</u> $Cl(k)$ [aka Picard group $Pic(k)$].

<u>Prop^n</u> If k'/k a finite ext^n, $\exists!$ $\mathcal{D}_k \to \mathcal{D}_{k'}$ s.t.

$$\begin{tikzcd} k^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}_k \arrow[d] \\ k'^\times \arrow[r] & \mathcal{D}_{k'} \end{tikzcd}.$$

So for K a poss. infinite global field, put

$$\mathcal{D}_K = \operatorname*{colim}_{\substack{k \subset K \\ \text{finite global field}}} \mathcal{D}_k$$

$$Cl(K) = \operatorname*{colim}_{\substack{k \subset K \\ \text{finite}}} Cl(k).$$

<u>Formations</u>

For a finite local or global field k define a locally cpct. ab. grp $A(k)$, called the group of (class) <u>formations</u>.

If k is local, $A(k) = k^\times$.

Now let k be global.

Def^n The idele group $\mathbb{J}(k) \hookrightarrow \prod_v k_v^\times$

consists of those families which are units at all but finitely many places.

i.e. $\in \mathcal{O}_v^\times$

NB: $\mathcal{O}_v^\times = k_v^\times$ if v is infinite.

Give $\mathbb{J}(k)$ the induced topology.

(Tychonov's theorem $\Rightarrow \mathbb{J}(k)$ locally cpct.)

Have $\mathbb{J}(k) \longrightarrow \mathcal{D}_k$.

$$\prod_v a_v \longmapsto \sum_{v \text{ finite}} v(a_v) \cdot v$$

Let K/k be a finite ext^n.

Obtain $\mathbb{J}(k) \hookrightarrow \mathbb{J}(K)$

$$\prod_v a_v \longmapsto \prod_w a_{w|_k}$$

If K/k is Galois then $\mathbb{J}(K)^{\mathrm{Gal}(K/k)} = \mathbb{J}(k)$.

Have $k^\times \hookrightarrow \mathbb{J}(k)$ (diagonally).

$A(k) := \frac{\mathbb{J}(k)}{k^\times}$ idele class group.

Lemma If K/k finite, then $A(k) \hookrightarrow A(K)$.
If K/k Galois, then $A(K)^{\mathrm{Gal}(K/k)} = A(k)$.

Pf Only prove 2^{nd} statement. $G = \mathrm{Gal}(K/k)$

$$1 \longrightarrow K^\times \longrightarrow J(K) \longrightarrow A(K) \longrightarrow 0$$

$$\begin{tikzcd}
\rightsquigarrow (K^\times)^G \arrow[r] \arrow[d, equal] & J(K)^G \arrow[r] \arrow[d, equal] & A(K)^G \arrow[r] \arrow[d, equal] & H^1(G, K^\times) \arrow[r] \arrow[d, equal] & \cdots \\
k^\times & J(k) & \therefore A(k) & 0 \text{ by Hilbert 90} &
\end{tikzcd}$$

$\square$

Let k be a finite local or global field, L/k a finite sep. ext^n, $K/L/k$, K/k finite Galois.

Define for $a \in A(L)$

$$N_{L/k}(a) = \prod_{\substack{L \overset{\alpha}{\hookrightarrow} K \\ k\text{-alg. hom}^{ns}}} \alpha(a) \in A(K)^{\mathrm{Gal}(K/k)} \cong A(k).$$

Obtain hom^n $N_{L/k} : A(L) \longrightarrow A(k)$.

One may show indep. of choice of K.

Class field theory

Th^m Let k be a finite local or global field. There is a bijective correspondence

$$\{\text{finite idx closed subgps of } A(k)\} \qquad A(k) \supset U = N_{K/k}(A(K))$$
$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$
$$\{\text{finite abelian ext}^{\text{ns}} \text{ of } k\} \qquad\qquad K/k$$

Given K/k finite abelian one has the reciprocity map

$$A(k) \longrightarrow \mathrm{Gal}(K/k), \quad \alpha \mapsto (\alpha, K/k)$$

with kernel U. "Artin symbol"

Rmk The Artin symbol has many good properties. E.g. if w is a finite place of K, $v = w|_k$ then

$$\begin{tikzcd} A(k_v) \arrow[r, hook] \arrow[d] & A(k) \arrow[d] \\ \mathrm{Gal}(K_w/k_v) \arrow[r] & \mathrm{Gal}(K/k) \end{tikzcd}$$

inclusion at v

Commute.

reciprocity maps

<u>Cor</u> Let k^{ab}/k be a max. abelian ext^n.
Then $\mathrm{Gal}(k^{ab}/k) \cong A(k)^{\wedge}$.
$(= \mathrm{Gal}(k)^{ab})$

prof. completion, i.e.
$\lim\limits_{\substack{U \subset A(k) \\ \text{normal} \\ \text{closed} \\ \text{finite idx}}} A(k)/U$

Cohomology of formations

<u>Th^m</u> k finite local or global field, K/k Galois.
Then
$$H^1(\mathrm{Gal}(K/k), A(K)) = 0$$
$$H^3(\mathrm{Gal}(K/k), A(K)) = 0$$
$$H^2(\mathrm{Gal}(K/k), A(K)) \xhookrightarrow{\mathrm{inv}_k} \mathbb{Q}/\mathbb{Z}$$
If K/k is finite then $H^2(\mathrm{Gal}(K/k), A(K))$ is a finite cyclic group of order $[K:k]$.

<u>Rmk</u> inv_k is related to the Artin symbol.

$\rightsquigarrow$ may relate properties of of Artin Symbol to properties of inv_k.

Cohomology of mult. gp (again)

Let K/k be a Galois ext^n, k a finite global field. For a place v of k, pick an extension w to K. $\rightsquigarrow$

$$\begin{tikzcd} H^2(\mathrm{Gal}(K/k), K^\times) \arrow[r] & H^2(\mathrm{Gal}(K_w/k_v), K_w^\times) \arrow[d, "\mathrm{inv}"] \\ & \mathbb{Q}/\mathbb{Z}. \end{tikzcd}$$

$$H^2(\mathrm{Gal}(K_w/k_v), K_w^\times) = A(K_w)$$

Th^m (Hasse local global principle)

The sequence

well-defined!

$$0 \to H^2(\mathrm{Gal}(K/k), K^\times) \longrightarrow \bigoplus_v H^2(\mathrm{Gal}(K_w/k_v), K_w^\times) \xrightarrow{\Sigma\, \mathrm{inv}} \mathbb{Q}/\mathbb{Z}$$

also infinite places!

is exact.

NB: $H^2(\mathrm{Gal}(k), \bar{k}^\times)$ is a.k.a. the Brauer group.

$H^2(\mathrm{Gal}(\mathbb{C}), \mathbb{C}^\times) = 0$

$$H^2(\mathrm{Gal}(\mathbb{R}), \mathbb{C}^\times) = \mathbb{Z}/2$$
$$H^2(\mathrm{Gal}(k_v), k_v^\times) = \mathbb{Q}/\mathbb{Z}.$$

Thm If K/k is finite then $H^3(\mathrm{Gal}(K/k), K^\times)$ is cyclic of order

$$\frac{[K:k]}{\mathrm{lcm}\,[K_w:k_v]}.$$