

Lecture 10: Class field theory

Frobenius

Th^m K/k Galois, v a finite place of K .

$$D_v \longrightarrow \text{Gal}(K(v)/k(v))$$

\uparrow
residue fields

is a surjection with kernel I_v .

PF omitted.

Now if $k(v)$ is finite, then $\hat{\mathbb{Z}} \longrightarrow \text{Gal}(K(v)/k(v)$
 $1 \longmapsto \text{Frobenius}$
 $x \longmapsto x^{|k(v)|}$

If v is unramified (i.e. $I_v = \text{Seq}$) then

$$D_v \cong \text{Gal}(K(v)/k(v)$$

$\downarrow \qquad \downarrow$
 $\sigma_v \qquad \text{Frobenius}$

'Frobenius element'

Lemma If K/k is Galois and unramified at v , & $K/L/k$ Galois
 $w = v|_L$. Then $\sigma_v|_L = \sigma_w$.

PF clear. \square

Local fields & global fields

A global field is an algebraic extension of \mathbb{Q} or $\mathbb{F}_p(x)$. It is called finite global field if the extⁿ is finite. A local field is an algebraic extⁿ of $\mathbb{Q}_p (= \text{Frac}(\hat{\mathbb{Z}}_p))$ or $\mathbb{F}_p((x))$. Finite local field if extⁿ is finite.

NB: K a global field, v a finite place,
then K_v is a local field.

\therefore often discuss \mathbb{R}, \mathbb{C} along with local fields

In this lecture: all fields local or global.

Class groups

Let k be a finite global field.

Defⁿ $\mathcal{D}_k =$ group of divisors in \mathcal{O}_k
 $=$ free abelian group on finite places

Have a hom^m $k^\times \longrightarrow \mathcal{D}_k$
 $x \longmapsto \sum_v v(x) v$

$\mathcal{O}_{k,v}$ is a DVR. Pick uni formizer π .

$$x = u \cdot \prod v(x)$$

The cokernel is called the (ideal) class group
 $Cl(K)$ [Picard group $Pic(K)$].

Prop³ If K'/K is finite extⁿ, $\exists!$ $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$

$$\begin{array}{ccc}
 K^\times & \longrightarrow & \mathcal{O}_K \\
 \downarrow & G & \downarrow \\
 K'^\times & \longrightarrow & \mathcal{O}_{K'}
 \end{array}$$

So for K a poss. infinite global field, put

$$\mathcal{O}_K = \text{cls}_{K \subset K} \mathcal{O}_K$$

finite global field

$$Cl(K) = \text{cls}_{K \subset K} Cl(K)$$

finite

Formations

For a finite local or global field K define
 a locally cap. ab. gp $A(K)$, called the group of
 (class) formations.

If K is local, $A(K) = K^\times$.

Now let k be global.

Defⁿ The idele group $J(k) \hookrightarrow \prod_v k_v^\times$

consists of those families which are units at all but finitely many places.

$$\begin{array}{c} \uparrow \\ \text{i.e. } \in \mathcal{O}_v^\times \\ \text{PB: } \mathcal{O}_v^\times = k_v^\times \text{ if } v \text{ is} \\ \text{finite.} \end{array}$$

Give $J(k)$ the induced topology.

(Tychonov's theorem $\Rightarrow J(k)$ locally comp.)

Have $J(k) \longrightarrow \mathcal{A}_k$.

$$\prod_v a_v \longmapsto \sum_{v \text{ finite}} v(a_v) \cdot v$$

Let K/k be a finite extⁿ.

Obtain $J(k) \hookrightarrow J(K)$

$$\prod_v a_v \longmapsto \prod_w a_{v|_K}$$

If k/k is Galois then $J(k)^{\text{Gal}(K/k)} = J(k)$.

Have $k^\times \hookrightarrow J(k)$ (diagonally).

$$A(k) := \frac{J(k)}{k^\times} \quad \underline{\text{idele class group.}}$$

Lemma If K/k finite, then $A(k) \xrightarrow{\text{Gal}(K/k)} A(K)$.

If K/k Galois, then $A(K)^{\text{Gal}(K/k)} = A(k)$.

Pf Only prove 2nd statement.

$$G = \text{Gal}(K/k)$$

$$1 \rightarrow K^* \rightarrow F(K) \rightarrow A(K) \rightarrow 0$$

$$\rightsquigarrow \begin{array}{ccccc} (K^*)^G & \longrightarrow & F(K)^G & \longrightarrow & A(K)^G \\ \parallel & & \parallel & & \parallel \\ K^* & & F(k) & \xrightarrow{\cong} & A(k) \end{array}$$

$$\hookrightarrow H^2(G, K^*) \rightarrow \dots$$

$\cong 0$ by Hilbert 90

□

Let k be a finite local or global field,

L/k a finite sep. extⁿ, $K/L/k$, K/k finite Galois.

Define for $a \in A(L)$

$$N_{L/k}(a) = \prod_{L \xrightarrow{\alpha} k} \alpha(a) \in A(K)^{\text{Gal}(K/k)} \cong A(k).$$

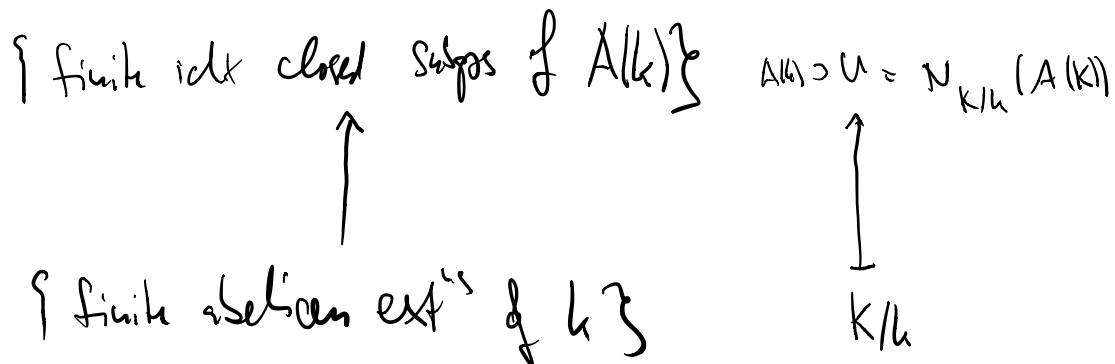
k -alg. hom^s

Obtain homⁿ $N_{L/k}: A(L) \rightarrow A(k)$.

One may show indep. of choice of K .

Class field theory

Th^m Let k be a finite local or global field. There is a bijection correspondence



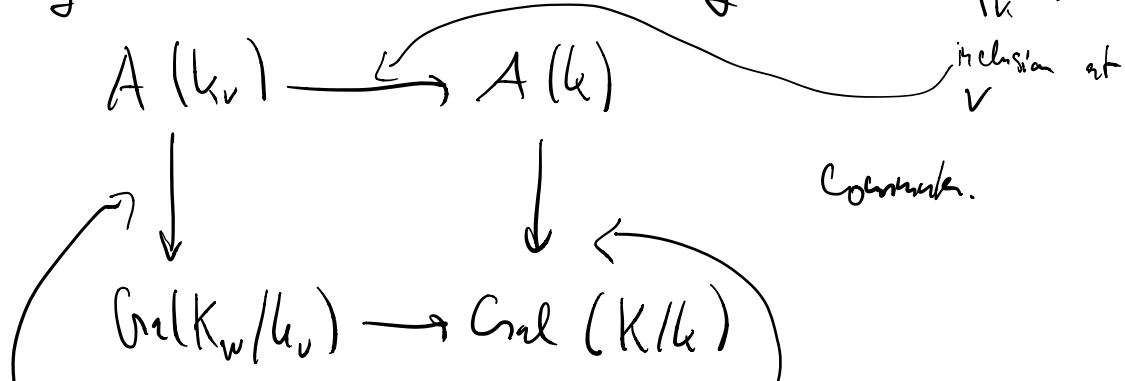
Given K/k finite abelian one has the reciprocity map

$$A(k) \rightarrow \text{Gal}(K/k), \quad \alpha \mapsto (\alpha, K/k)$$

with kernel U . "Artin symbol"

Prop The Artin symbol has many good properties.

E.g. if w is a finite place of K , $v = w|_k$ then



reciprocity
maps

Cor Let k^{as}/k be a max. abelian extⁿ.

Then $\text{Gal}(k^{as}/k) \cong A(k) \uparrow$
 (= $\text{Gal}(k^{as})$)

prof. completion, i.e.
 $\varprojlim_{U \subset A(k)} U$
 U normal closed
 finite idyl

Cohomology of formations

Th^m k finite local or global field, K/k Galois.

Then $H^2(\text{Gal}(K/k), A(K)) = 0$
 $H^3(\text{Gal}(K/k), A(K)) = 0$
 $H^2(\text{Gal}(K/k), A(K)) \xrightarrow{\text{inv}_k} \mathbb{Q}/\mathbb{Z}$

If K/k is finite then \uparrow is a finite cyclic group of order $[K:k]$.

Rank inv_k is related to the Artin symbol.

→ may relate properties of Artin symbol to properties of inv_v .

Cohomology of mult. gp (again)

Let K/k be a Galois extⁿ, k a finite global field. For a place v of k , pick an extension w to K .

$$\begin{array}{ccc} H^2(\text{Gal}(K/k), K^\times) & \longrightarrow & H^2(\text{Gal}(K_w/k_v), K_w^\times) \\ & & \downarrow \text{inv} \\ & & \mathbb{Q}/\mathbb{Z} \end{array}$$

\uparrow
 $A(K_w)$

Th^m (Hasse local global principle)

The sequence well-defined!

$$\begin{array}{ccc} 0 \rightarrow H^2(\text{Gal}(K/k), K^\times) & \longrightarrow & \bigoplus_v H^2(\text{Gal}(K_w/k_v), K_w^\times) \\ & & \downarrow \sum \text{inv} \\ & & \mathbb{Q}/\mathbb{Z} \end{array}$$

↗
also which place!

is exact.

NB: $H^2(\text{Gal}(k), k^\times)$ is a.k.a. the Brauer group.

$$H^2(\text{Gal}(\mathbb{C}), \mathbb{C}^\times) = 0$$

$$H^2(\text{Gal}(\mathbb{P}), \mathbb{C}^*) = \mathbb{Z}/2$$

$$H^2(\text{Gal}(k_v), k_v^*) = \mathbb{Q}/\mathbb{Z}.$$

Th If K/k is finite then

$H^3(\text{Gal}(K/k), K^*)$ is cyclic of order

$$\frac{[K:k]}{\text{lcm}[k_w:k_v]}.$$