Algebraic Number Theory

Exercises 5

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let $0 \neq I \subset A$ be an ideal, where A is a Dedekind domain. Prove that as an A-module, I is locally free of rank 1.

Deduce that if $I, J \subset A$ are two (arbitrary) ideals, then the canonical map $I \otimes_A J \to IJ$ is an isomorphism.

Exercise 2. Let A be any ring. Show that if $I, J \subset A$ are any two ideals such that $I + J = A$, then $I \cap J = IJ$. Deduce that

$$A/IJ \xrightarrow{\simeq} A/I \times A/J.$$

Exercise 3. Let A be a Dedekind domain and $f \in A$. Let S denote the multiplicative set $\{f^n \mid n \geq 0\}$.

(1) Show that the set $\{P \in \operatorname{Spec}(A) \mid P \cap S \neq \emptyset\}$ is finite. Denote the cardinality of this set by r.
(2) Show that $(S^{-1}A)^\times / A^\times$ is a free abelian group of rank $\leq r$. [*Hint:* use the exact sequence $0 \to A^\times \to K^\times \to \mathfrak{F}(A)$, and similarly for $S^{-1}A$.]

Exercise 4. Let A be a dvr with maximal ideal m and uniformizer π. Let $P(X) \in A[X]$ be a monic polynomial. Set $B = A[X]/P$.

(1) Suppose that the reduction of P modulo m is irreducible. Show that B is a dvr and π is a uniformizer of B.
(2) Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$. Suppose that a_0 is a uniformizer of A and $a_0 \mid a_i$ for all i. Show that B is a dvr and the class of X is a uniformizer of B. [*Hint:* use Eisenstein's criterion.]