Algebraic Number Theory

Exercises 4

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let $K = \mathbb{Q}(i)$, where $i^2 = -1$. Find all rational primes p such that the integral closure of $\mathbb{Z}_{(p)}$ in K is a dvr.

[*Hint:* use the decomposition of p in $\mathbb{Z}[i]$ established at the beginning of the lecture.]

Exercise 2. Let A be a noetherian domain and $f_1, \ldots, f_n \in A$ generating the unit ideal. Put $K = Frac(A)$.

(1) Show that $A = \cap_i A_{f_i}$ as subsets of K.
(2) Show that A is Dedekind if and only if A_{f_i} is Dedekind for every i.

Exercise 3. For $n > 0$, denote by $z_n \in \mathbb{C}$ a primitive n-th root of unity. Let p be an odd prime and $L = \mathbb{Q}(z_p)$ the corresponding cyclotomic field. Recall that $L/\mathbb{Q}$ is Galois with group $(\mathbb{Z}/p)^\times$.

(1) Deduce that $L/\mathbb{Q}$ has a unique quadratic subextension K.
(2) Using the formula $d_L = (-1)^{\frac{p-1}{2}} p^{p-2}$, find K explicitly. [*Hint:* Compute d_L in terms of the embeddings of L into an algebraic closure of $\mathbb{Q}$.]
(3) Deduce that any quadratic extension of $\mathbb{Q}$ embeds into a cyclotomic field. [*Hint:* Show that $\sqrt{2} \in \mathbb{Q}(z_8)$ and use that $\mathbb{Q}(z_n) \subset \mathbb{Q}(z_{mn})$.]