Algebraic Number Theory

Exercises 11

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let A a Dedekind domain with field of fractions K, L/K be a finite separable field extension and B the integral closure of A in L. Let $x_1, \ldots, x_n \in B$ such that $D_A^B(x_1, \ldots, x_n)$ generates the discriminant ideal $D_A^B \subset A$ (in particular this ideal is principal). Show that $x_1, \ldots, x_n$ form an A-basis of B.

[*Hint:* To show that $\bigoplus_i A\{x_i\} \to B$ is surjective, if y is outside the image, write $y = \sum_i \lambda_i x_i$ for $\lambda_i \in K$. Assuming WLOG that $\lambda_1 \notin A$, relating the bases $(y, x_2, \ldots, x_n)$ and $(x_1, \ldots, x_n)$, derive a contradiction.]

Exercise 2. Let p be an odd prime, ζ_p a primitive p-th root of unity, $K = \mathbb{Q}(\zeta_p)$. Show that $\mu(K) \simeq \mathbb{Z}/2p$.

[*Hint:* if $\zeta_q \in K$ for an odd prime q, then q ramifies in K.]

Exercise 3. Let p be a prime, $r \geq 1$, $p^r \geq 3$. Let $K_{p^r} = \mathbb{Q}(\zeta_{p^r})$, where ζ_{p^r} is a primitive p^r-th root of unity.

(1) How that $\mathcal{O}_{K_{p^r}} = \mathbb{Z}[\zeta_{p^r}]$.
(2) Show that $d_{K_{p^r}} = \pm p^s$ (for some integer s to be determined).