Algebraic Number Theory

Exercises 10

Dr. Tom Bachmann Winter Semester 2021/2

Exercise 1. Let $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free positive integer. Let $A = \mathcal{O}_K$.

(1) Show that
$$H := A^\times \cap (0, \infty) \subset A^\times$$
is isomorphic to $\mathbb{Z}$. Deduce that there is only one generator of H which is greater than 1. We call this generator the *fundamental unit*.

(2) Let $u = \inf\{v \in A^\times \mid v > 1\}$. Show that u is the fundamental unit of K.

(3) For every positive integer n, let $u^n = a_n + b_n\sqrt{d}$, where $a_n, b_n \in \mathbb{Q}$. Show that the sequence b_n is increasing.

(4) Find the fundamental unit for $d = 2, 7$. [*Hint: use (3) and the explicit description of A.*]

(5) Suppose that $d \not\equiv 1 \pmod 4$ and the fundamental unit u is known. Describe the solutions of the Pell–Fermat equations $x^2 - dy^2 = \pm 1$.

(6) Describe the positive integer solutions of the Pell–Fermat equations for $d = 2, 7$.

Exercise 2. Let $L = \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5-th root of unity.

(1) Show that $\mu(L) \simeq \mathbb{Z}/10$ and conclude that $\mathcal{O}_L^\times \simeq \mathbb{Z}/10 \oplus \mathbb{Z}$.

(2) Let $u \in \mathcal{O}_L^\times$. Show that $\bar{u}/u \in \mu(L)$.

(3) Show that if $\bar{u}/u = \lambda^2$ then $\bar{\lambda} = \lambda^{-1}$ and $u' = u\lambda \in \mathcal{O}_L^\times$ satisfies $\overline{u'} = u'$.

(4) Show that if $\bar{u}/u$ is *not* a square then u^5 is pure imaginary.

(5) Show that $\mathcal{O}_L^\times$ contains no pure imaginary elements, as follows. Set $\omega = \zeta + \zeta^{-1}$. Compute the minimum polynomial of ω and show that ω is a unit. Show that we may write any $u \in \mathcal{O}_K$ as $a + b\zeta$, with $a, b \in \mathbb{Z}[\omega]$. If u is a pure imaginary unit then show that $(1 - 2\zeta/\omega)$ is a (pure imaginary) unit. Show that this is false.

(6) Conclude that there exists $\alpha \in \mathcal{O}_L^\times \cap \mathbb{R}$ such that
$$\mu(L) \times \mathbb{Z} \to \mathcal{O}_L^\times, (\mu, n) \mapsto \mu \cdot \alpha^n$$
is an isomorphism.