Algebraic Number Theory

Exercises Tutorium 9

Dr. Tom Bachmann — Winter Semester 2021–22

Let $x^3 = 2$ and $K = \mathbb{Q}(x)$. The aim of these exercises is to determine $\mathcal{O}_K$.

Exercise 1. Show that $X^3 - 2$ is irreducible in $\mathbb{Q}[X]$ and $[K : \mathbb{Q}] = 3$.

Exercise 2. Let $z = a + bx + cx^2 \in K$. Compute $tr_{\mathbb{Q}}^{K}(z)$.

Exercise 3. Let $A = \mathcal{O}_K$ and $B \subset K$ the subring $\mathbb{Z}[x]$ generated by x. Show that $B \subset A$ and B is a free abelian group with basis $\{1, x, x^2\}$.

Exercise 4. Show that xA is a prime ideal in A [*hint:* consider the decomposition of $2A$ into prime ideals] and xB a prime ideal in B. What is the residue field of xA? Show that $B/xB \simeq A/xA$. Conclude that $A = B + xA$ and $A = B + 2A$.

Exercise 5. Show that $3 = (x - 1)(x + 1)^3$ and $x - 1$ is a unit in B. Proceeding as in (5), with $1 + x$ in place of x, 3 in place of 2, show that $A = B + (x + 1)A$ and $A = B + 3A$.

Exercise 6. Conclude that $A = B$.