Algebraic Number Theory

Exercises Tutorium 8

Dr. Tom Bachmann Winter Semester 2021–22

Exercise 1. Let A be a Dedekind domain, P a maximal ideal such that A/P is finite. Show that

$$|A/P^e| = |A/P|^e.$$

Exercise 2. Let $K/\mathbb{Q}$ be a number field and I a non-zero ideal. Suppose that

$$I = \prod_i P_i^{e_i}$$

is the factorization of I in $F(\mathcal{O}_K)$, and $P_i \cap \mathbb{Z} = (p_i)$ with $p_i > 0$. Show that

$$N_{\mathbb{Q}}^K(I) = \prod_i p_i^{e_i[\mathcal{O}_K/P_i:\mathbb{F}_{p_i}]}.$$

Taking $I = (p)$, deduce that

$$[K : \mathbb{Q}] = \sum_i e_i[\mathcal{O}_K/P_i : \mathbb{F}_p].$$

Exercise 3. Let L/K be finite Galois with group G. Let $A \subset K$ be integrally closed, with integral closure B in L. Let P be a prime ideal of A. Show that G permutes the prime ideals of B lying over P transitively.

[*Hint:* you may wish to recall or prove first the following facts: (1) if an ideal I of A is contained in a finite union of prime ideals, then I is contained in one of the ideals, and (2) in any integral extension, there are no proper inclusions of primes above the same prime.]