Algebraic Number Theory

Exercises Tutorium 7

Dr. Tom Bachmann — Winter Semester 2021–22

Exercise 1. Prove the 5-lemma: given a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

in which a, b, d, e are isomorphisms, then also c is an isomorphism. [Can you weaken the hypotheses?]

Exercise 2. Suppose given a commutative diagram with exact rows as follows

$$\begin{array}{ccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

(1) Construct a natural map $\partial : ker(c) \to cok(a)$. Give an example when this map is non-zero.

(2) Prove the snake lemma: The sequence

$$ker(a) \to ker(b) \to ker(c) \xrightarrow{\partial} cok(a) \to cok(b) \to cok(c)$$

is exact. [Which further hypotheses ensure exactness at the beginning and end?]

Exercise 3. Let $M : \mathbb{Z}^n \to \mathbb{Z}^n$ be a homomorphism. Show that $\mathbb{Z}^n/M\mathbb{Z}^n$ is finite if and only if $det(M) \neq 0$, and in this case $|\mathbb{Z}^n/M\mathbb{Z}^n| = det(M)$.

[*Hint:* Smith normal form.]