Algebraic Number Theory

Exercises Tutorium 6

Dr. Tom Bachmann — Winter Semester 2021–22

Exercise 1. Let A be a Dedekind domain and $0 \neq I \subset A$ an ideal. Show that A/I is a principal ideal ring. [*Hint*: use the strong approximation theorem from Exercise 4 of Sheet 6.]

Deduce that I can be generated by 2 elements.

Exercise 2. Let $0 \neq I, J$ be ideals in a Dedekind domain. We show that there exists I' in the same ideal class as I, and coprime to J, as follows. Pick $0 \neq a_0 \in I$. Show that there exists an ideal K such that $IK = (a_0)$. Show that $K = JK + (x_0)$. Deduce that $(1) = J + Ix_0/a_0$, and conclude.

Exercise 3. Let $0 \neq I, J$ be ideals in a Dedekind domain A. Show that $I \oplus J \simeq A \oplus IJ$. [*Hint*: consider first the case $I + J = A$.]

Recall from last week that any ideal of A is locally free, hence finite projective as an A-module.

Exercise 4. Let A be a Dedekind domain. Show that any finite projective A-module is isomorphic to a direct sum of ideals.

Exercise 5. Let A be a Dedekind domain and $I_1, \dots, I_s,\ J_1, \dots, J_r$ non-zero ideals. Show that

$$I_1 \oplus \cdots \oplus I_s \simeq J_1 \oplus \cdots \oplus J_r$$

if and only if $s = r$ and $\prod_i I_i$, $\prod_j J_j$ lie in the same ideal class.