Algebraic Number Theory

Exercises Tutorium 4

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Recall the definition of a multiplicatively closed subset $S \subset A$ of a commutative ring A, and the localization $S^{-1}A$.

Exercise 1. Show that $P \mapsto S^{-1}P := P \cdot S^{-1}A$ induces an inclusion preserving bijection between prime ideals of A not meeting S and prime ideals of $S^{-1}A$. What about maximal ideals (respectively all ideals)?

Exercise 2. Let A be a ring and $P \subset A$ a prime ideal. Let $S_P = A \setminus P$. Show that S_P is multiplicative. Put $A_P = S_P^{-1}A$. Show that A_P has a unique maximal ideal, namely $S_P^{-1}P =: P_P$. Show also that A_P/P_P is the fraction field of the domain A/P.

Exercise 3. Let A be a commutative ring. Show that the following are equivalent:

(1) A has a unique maximal ideal.
(2) $A \setminus A^{\times}$ is an ideal.
(3) $A \setminus A^{\times}$ is the unique maximal ideal of A.

(Such a ring is called *local.*)

Exercise 4. *Extra:* show that $S^{-1}A$ is a flat A-module.