Algebraic Number Theory

Exercises Tutorium 3

Dr. Tom Bachmann Winter Semester 2021–22

Exercise 1. Prove Dedekind's Lemma: If $\chi_1, \ldots, \chi_n : G \to K^\times$ are distinct homomorphisms from a group G to the multiplicative group of a field, and if $a_1, \ldots, a_n \in K$ such that $\sum_i a_i \chi_i : G \to K$ is the zero map, then $a_i = 0$ for all i.

Exercise 2. Using Dedekind's Lemma, show that if E, F are extensions of a field K, and $[E : K] = n$, then there are at most n k-algebra homomorphisms from E to F.

Exercise 3. Let A be a ring and $a_1, \ldots, a_n \in A$. Consider the $n \times n$ Vandermonde matrix M with entries $M_{ij} = a_i^{j-1}$.

(1) Show that if $a_i = a_j$ for some $i \neq j$, then $|M| = 0$.

(2) Suppose that $A = \mathbb{Z}[X_1, \ldots, X_n]$ and $a_i = X_i$. Show that $(X_j - X_i)$ divides $|M|$ for $i \neq j$. Deduce that

$$|M| = \prod_{1 \leq i < j \leq n} (X_i - X_j).$$

(3) Deduce that for $(A, a_1, \ldots, a_n)$ general we have

$$|M| = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

Exercise 4. Show that given $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ with the a_i distinct, there exists a unique monic polynomial $P(X) \in A[X]$ of degree n such that $P(a_i) = b_i$.