Algebraic Number Theory
Exercises Tutorium 2

Dr. Tom Bachmann — Winter Semester 2021–22

For a positive integer n, define the n-th *cyclotomic polynomial* as

$$\Phi_n(X) = \prod_{\zeta \in W_n} (X - \zeta).$$

Here $W_n \subset \mathbb{C}$ denotes the set of primitive n-th roots of unity.

Exercise 1. Show that $\Phi_n(X) \in \mathbb{Z}[X]$.

Exercise 2. Show that

$$X^n - 1 = \prod_{d|n} \Phi_d(X) \in \mathbb{Z}[X].$$

Exercise 3. Show that

(1) If $n = p$ is prime, then

$$\Phi_p(X) = 1 + X + \cdots + X^{p-1}.$$

(2) If $n = 2m$, $m > 1$ odd, then

$$\Phi_n(X) = \Phi_m(-X).$$

(3) For $n > 2$, $\Phi_n(X)$ is palindromic (its list of coefficients reads the same forward and backwards).

Exercise 4. Let k be any field and $x \in k$ a primitive n-th root of unity. Show that $\Phi_n(x) = 0 \in k$.

Extra: What about the converse?

Exercise 5. Show that $\Phi_5(X)$ is irreducible over $\mathbb{F}_2$.

Hint: What would the existence of a quadratic factor tell you about roots of unity in $\mathbb{F}_4$?