Algebraic Number Theory

Exercises Tutorium 12

Dr. Tom Bachmann Winter Semester 2021–22

Let A be a Dedekind domain with field of fractions K, L/K a finite separable extension, B the integral closure of A in L.

Exercise 1. Let

$$B^\vee := \{x \in L \mid tr_K^L(xy) \in A \text{ for all } y \in \mathcal{B}\}.$$

(1) Suppose that B admits an integral A-basis $x_1, \ldots, x_n$. Show that $B^\vee$ admits an A-basis $y_1, \ldots, y_n$ characterized by $tr(x_i y_j) = \delta_{ij}$.
(2) Show that $B^\vee$ is a non-zero fractional ideal.
(3) Let $Diff_A^B = (B^\vee)^{-1}$. Show that $Diff_A^B$ is an integral ideal.

From now on let L/K be Galois.

Exercise 2. Show that $D_A^B(B^\vee) \cdot D_A^B = 1$ and $N_K^L(Diff_A^B) = D_A^B$.
[*Hint:* reduce to the case where B admits an integral A-basis and is a PID.]

Exercise 3. Show that a prime P of B is ramified if and only if $Diff_A^B \subset P$.

Exercise 4. Show that $\mathcal{O}_{K(\sqrt[3]{2})}$ is a PID.