Algebraic Number Theory

Exercises Tutorium 11

Dr. Tom Bachmann Winter Semester 2021–22

Exercise 1. Let L/K be a finite separable extension.

(1) Show that $N^L_K(x) \prod_{i=1}^{[L:K]} x_i$, where $x_1, \ldots, x_n$ are the roots of the minimal polynomial of x, each repeated $[L : K(x)]$ times.

(2) Suppose L/K is Galois with group G. Show that

$$N^L_K(x) = \prod_{g \in G} gx.$$

Exercise 2. Let $A \subset B \subset C$ be finite extension of Dedekind domains, and $P \subset A$ a prime ideal. Show that if P ramifies in B it also ramifies in C.

Exercise 3. Let A be a ring and M a A-module. Write $\Lambda^n M$ for the quotient of $M^{\otimes n}$ by the submodule generated by elements of the form $m_1 \otimes \cdots \otimes m_n$, where $m_i = m_j$ for some $i \neq j$. Write $m_1 \wedge \cdots \wedge m_n \in \Lambda^n M$ for the image of $m_1 \otimes \cdots \otimes m_n$.

(1) Show that

$$m_1 \wedge \cdots \wedge m_n = -m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge m_i \wedge m_{i+2} \wedge \cdots \wedge m_n.$$

(2) Show that

$$\Lambda^n A^m \simeq A^{\binom{m}{n}}.$$

Exercise 4. Let A be a Dedekind domain.

(1) Suppose that P is a locally free A-module of rank n. Show that P is free if and only if $\Lambda^n P$ is free. [*Hint*: recall Tutorium 6.]

(2) Let $L/K = Frac(A)$ be a finite separable extension, and B the integral closure of A in L. Suppose that $D^B_A \subset A$ is principal. Suppose further that $Pic(A)$ has no 2-torsion. Show that B is free as an A-module.