

## Sheet 7

1.  $K/k$  f.t., transc. deg. 1,  $k \subset R \subset K$  val<sup>y</sup> ring ( $\text{Frac } R = K$ ).  
Find Ded. dom.  $k \subset A \subset R$  f.t.  $k$  &  $P \in \text{Spec } A$  with  
 $A_P = R$ .

Let  $K/k$  be gen. by  $x_1, \dots, x_n$ . Replacing  $x_i$  by  $x_i^{-1}$   
if necessary,  $\forall i$   $x_i \in R$ . Let  $A' = k[x_1, \dots, x_n] \subset R$   
 $A = \text{normalization}$ .

Then  $k \subset A \subset R$ .  
f.t.

Let  $P = A \cap \mathfrak{m}_P$ . Then  $P$  is a prime ideal  $\neq 0$

(otherwise  $\text{Frac}(A) \subset R$  ~~is~~).  $\therefore \text{ht}(P) = 1$

$A_P$  is a DVR

$\forall \mathfrak{m} \subset A_P \subset R$  is dominations of val<sup>y</sup> rings.

$\therefore A_P = R \quad \square$

2.  $C/k$  proper regular curve

Show  $\{ \varphi: C \rightarrow \mathbb{P}_k^1 \text{ bir. surj.} \}$

$\cong \{ f \in k(C) \text{ transc. } /k \}$ .

Given  $\varphi: C \rightarrow \mathbb{P}^2$  fin. seg. other

$$\varphi_0: C_0 \rightarrow A^2 \text{ --- " ---}$$

$\downarrow$   
 $\text{Spec } A$

$$\therefore \varphi_0^\# : k(C) \rightarrow A \text{ finite \& injective}$$

$\uparrow \quad \longmapsto \quad f$

Thus  $f \in k(C) = \text{Frac}(A)$  is transcendental.

Conversely  $f \in k(C)$  transcendental yields

$$C \supset C_0 \xrightarrow{f} A^1 \subset \mathbb{P}^1.$$

Extend to  $C$  by ex. 4 last sheet.

Then  $\varphi: C \rightarrow \mathbb{P}^2$  is proper \& non-const.

$\therefore$  finite seg. by ex 3 last sheet.

clear these constructions are invert.  $\square$

3.  $X, Y$  proper regular curves  $k$ .

$$\text{Show: } k(X) \subseteq k(Y) \Rightarrow X \cong Y.$$

$$k(X) \xrightarrow{\cong} k(Y) \quad \text{Spec } A \subset \text{Spec } B \text{ open aff.}$$

$U$   
 $A$

$U$   
 $B$

$\text{Spec } B \subset Y$

We have  $A \xrightarrow{f \cdot \frac{1}{f}}$   $k(Y) = \text{Frac}(B)$

$\therefore$  also  $A \xrightarrow{f}$   $B_f$  for some  $f \in B$

Hence: find  $U \subset Y$  open &  $U \xrightarrow{f} X$   
a birat. iso.

Extend to  $Y$  by prev. week.

We have birat. iso  $Y \xrightarrow{f} X$ .

$\therefore$  open im.  $\hookrightarrow$  ex 2 last sheet.

$\therefore$  iso since proper

4.  $X$  affine curve/ $k$ .  $f: X \rightarrow X$  closed.  
Show:  $X \times X$  affine.

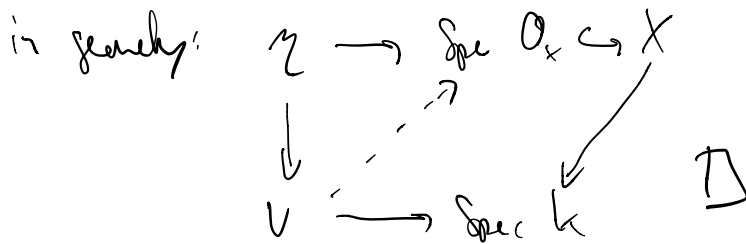
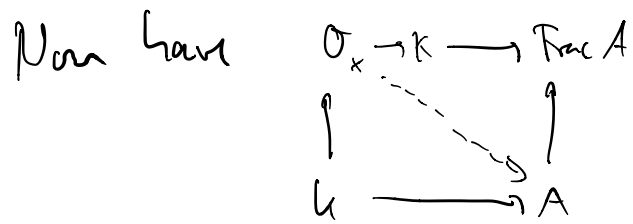
Show "instead" that  $X \times X \rightarrow X$  is an affine morphism. This property is local on the target!

Hence: we may replace  $X$  by any affine open  
sub. of  $X$ .

Case 1:  $x \in X$  a regular pt. Then the defining ideal



$\therefore A \cap k = \mathcal{O}_x$  for some  $x \in X_U$ .



2. Show that categories

$\mathcal{C}_1 = \{ \text{curves } / k \text{ \& nat}^l \text{ dominant morphisms} \}$   
 &

$\mathcal{C}_2 = \{ \text{ft. field ext. } / k \text{ of br. dg. } 1 \}$

are equiv.

Functors  $\mathcal{C}_2 \rightarrow \mathcal{C}_1$   
 $k \mapsto X_k$

$\mathcal{C}_1 \rightarrow \mathcal{C}_2$   
 $X \mapsto k(X)$ .

Then  $k(X_k) \cong k$  is clear.

If  $X \in \mathcal{C}_1$  then obtain  $X \xleftarrow{\circ} X_{k(X)}$

↙  
Singular iso.

Check there is a natural.  $\square$