Sheet 7

1. K/k f.t., transc. deg. 1, $k \subset R \subset K$ val^n ring (Frac $R = K$).
Find Ded. dom. $k \subset A \subset R$ f.t./k & $\mathfrak{p} \in \operatorname{Spec} A$ with $A_\mathfrak{p} = R$.

Let K/k be gen. by $x_1, \dots, x_n$. Replacing x_i by x_i^{-1} if necessary, WMA $x_i \in R$. Let $A' = k[x_1, \dots, x_n] \subset R$
A = normalization.

Then $k \underset{\text{f.t.}}{\subset} A \subset R$.
Let $\mathfrak{p} = A \cap \mathfrak{m}_R$. Then $\mathfrak{p}$ is a prime ideal $\neq 0$ (otherwise $\underset{K}{\underset{\shortparallel}{\operatorname{Frac}}}(A) \subset R$ $\cancel{\%}$). $\therefore$ ht$(\mathfrak{p}) = 1$
$A_\mathfrak{p}$ is a DVR

Now $A_\mathfrak{p} \subset R$ is domination of val^n rings.
$\therefore A_\mathfrak{p} = R$ $\square$

2. C/k proper regular curve.
Show $\{\varphi : C \to \mathbb{P}^1_k \text{ fin. surj.}\}$
$\simeq \{f \in k(C) \text{ transc. } /k\}$.

Given $\varphi : C \to \mathbb{P}^1$ fin. surj: obtain

$$\varphi_0 : \underset{\substack{\shortparallel \\ \operatorname{Spec} A}}{C_0} \to \mathbb{A}^1 \quad - \;\shortparallel\; -$$

$\therefore$ $\varphi_0^\# : k[T] \to A$ finite & injective.
$T \longmapsto f$

Thus $f \in k(C) = \operatorname{Frac}(A)$ is transcendental.

Conversely $f \in k(C)$ transcendental yields

$$C \supset C_0 \xrightarrow{f} \mathbb{A}^1 \subset \mathbb{P}^1.$$

Extend to C by ex. 4 last sheet.

Then $\varphi : C \to \mathbb{P}^1$ is proper & non-const.

$\therefore$ finite surj. by ex 3 last sheet.

Clearly these constructions are inverse. $\square$

3. X, Y proper regular curves $/k$.
Show: $k(X) \cong k(Y) \Rightarrow X \cong Y$.

$K(X) \xrightarrow{\cong} K(Y)$ $\qquad$ $\operatorname{Spec} A \subset X$ open aff.

$\cup \qquad\qquad \cup \qquad\qquad \operatorname{Spec} B \subset Y$

$A \qquad\qquad B$

Now have $\underset{\text{f.t./k}}{A} \longrightarrow k(Y) = \operatorname{Frac}(B)$

$\therefore$ obtain $A \longrightarrow B_f$ for some $f \in B$

Hence: find $U \subset Y$ open & $U \longrightarrow X$

a birat. iso.

Extend to Y by prev. week,

Now have birat. iso $Y \longrightarrow X$.

$\therefore$ Open imm. by ex 2 last sheet.

$\therefore$ iso since proper

4. X affine curve$/k$. $x \in X$ closed.
Show: $X \setminus x$ affine.

Show "instead" that $X \setminus x \longrightarrow X$ is an affine morphism. This property is local on the target!
Hence: we may replace X by any affine open nbhd. of x.

Case I: $x \in X$ a regular pt. Then the defining ideal is locally principal. $\therefore$ WMA $X = \operatorname{Spec} A$, $m_x = (a)$.

Then $X \setminus x = \operatorname{Spec} A[\tfrac{1}{a}]$.

Case 2: $x \in X$ singular. WMA: only singular pt.

Normalization $\begin{array}{c}\tilde{X}\\ \downarrow\\ X\end{array}$. Then $X \setminus x = \tilde{X} \setminus \underbrace{\{\text{fiber over } x\}}_{\text{finitely many closed pts}}$.

$\therefore$ $X \setminus x$ affine by repeated appl^n of Case 1.

$\square$

Sheet 8

1. K/k f.t. tr.deg. 1, abstract regular curve X_K. Show that X_K is proper using valuative crit^n.

$$\begin{tikzcd}
 & & \operatorname{Spec} K \arrow[d, hook] \\
\eta \arrow[urr] \arrow[rr] \arrow[d] & & X_K \arrow[d] \\
V \arrow[rr] \arrow[urr, dashed, "?"] & & \operatorname{Spec} k
\end{tikzcd}$$

Spec of val^n ring

in algebra:

$$\begin{tikzcd}
\text{“}?\text{''} \arrow[r] & K \arrow[r] & \operatorname{Frac}(A) \\
k \arrow[u] \arrow[ur] \arrow[rr] & & A \arrow[u]
\end{tikzcd}$$

Now $K \cap A$ is a val^n ring of K c/t. k.

$\therefore$ done by ex 1 prev. sheet

$\therefore\ A \cap K = \mathcal{O}_x$ for some $x \in X_k$.

Now have
$$\begin{tikzcd} \mathcal{O}_x \arrow[r] \arrow[drr, dashed] & K \arrow[r] & \operatorname{Frac} A \\ k \arrow[u] \arrow[rr] & & A \arrow[u] \end{tikzcd}$$

in geometry:
$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_x \arrow[r, hook] & X \arrow[dl] \\ V \arrow[ur, dashed] \arrow[r] & \operatorname{Spec} k & \end{tikzcd}$$

$\square$

2. Show that categories

$\mathcal{C}_1 = \{$curves $/k$ & rat^l dominant morphisms$\}$

&

$\mathcal{C}_2 = \{$f.t. field ext. $/k$ of tr.deg. $1\}$

are equiv.

Functors $\mathcal{C}_2 \longrightarrow \mathcal{C}_1$ $\qquad$ $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$

$K \longmapsto X_K$ $\qquad$ $X \longmapsto k(X)$.

Then $k(X_K) \cong K$ is clear.

If $X \in \mathcal{C}_1$ then obtain $X \hookleftarrow X_{k(X)}$

bidirectional iso.

Check these iso are natural. ∎