

1. A integral domain is of finite type over field k .

Show: A is integrally closed \Leftrightarrow if $P \in \text{Spec } A$ has $\text{ht } 1$, then A_P is a DVR, & $A = \bigcap_{\text{ht}(P)=1} A_P$.

\Leftarrow : A_P DVR $\Rightarrow A_P$ i.c. $\Rightarrow \bigcap_{A_P} A_P$ i.c.
 $\parallel \text{localization}$

\Rightarrow : A i.c. $\Rightarrow S^{-1}A$ i.c.

$$\left[T^n + \frac{a_{n-1}}{s} T^{n-1} + \dots + \frac{a_0}{s^n} = 0 \text{ for some } T \in \text{Traca} \right]$$

Let $S = \{T^i s\}$, $\text{ht } 1 \hookrightarrow S^{-1}A \Rightarrow ST$ is integral over A

$$\therefore ST \in A \quad \therefore T \in S^{-1}A$$

$$\therefore A_P \text{ i.c. } \forall P \in \text{Spec } A$$

Need to show: $A = \bigcap_{\text{ht } P=1} A_P$.

Use Noether normalization: $\exists A_0 = k[x_1, \dots, x_n] \xrightarrow{\text{finite}} A$.

First prove for A_0 , which is a UFD:

- if $\pi \in A_0$ is a prime elt, then (π) is a prime ideal of $\text{ht } 1$. (prime: clear. $\text{ht } 1$: $P \subseteq (\pi)$,

$$\text{if } f \in P, f = \pi^h \cdot g, \pi \nmid g.$$

B/c P prime, $f \in \mathbb{N}$ or $\pi \in \mathbb{P}$ (reduced $\frac{f}{\pi}$)

- $A_{(p)} = \frac{\text{fractions w/ denom. not divisible by } p}{p}$
- $\therefore \bigcap_{\pi} A_{(p)} = \text{fractions with denominator 1, unit} = A.$

For A : $x \in L = \text{Frac}(A)$

$L/K = \text{Frac}(A_0)$ is a finite extⁿ.

Note that A is the integral closure of A_0 in L .

$P(T) \in K(T)$ monic min poly of x .

Want to show that P has coeff in A_0 . (Then x/A_0 int. $\therefore x \in A$.)

STP P has coeff in $(A_0)_Q$, $Q \in \text{Spec } A_0$ has $\text{ht } 1$.

May replace A by A_Q .

Now $\dim A = 1$. $\therefore \{\text{prims of ht } = 1\} = \{\text{max ideals}\}$.

General fact: if A is an i.d., then $A = \bigcap_{\text{max}} A_{\text{max}}$.

This concludes.

Proof of fact: $x \in \text{Frac}(A)$.

$$I = \{a \in A \mid x \cdot a \in A\}$$

want: $1 \in I$. If $x \in A_m \Rightarrow \exists \frac{a}{s} \in A \setminus m$ s.t. $x = \frac{a}{s}$

$\therefore s \in I \quad \therefore I \not\subseteq m$

$\therefore 1 \in I$ as desired. \square

2. $f: C \rightarrow D$ ^{sur.} map. of curves & D is regular.
 Show: f is an open immersion.

Step 1: for $x \in C$, $\mathcal{O}_{D, f(x)} \xrightarrow{\cong} \mathcal{O}_{C, x}$

WMA $D = \text{Spec } A, C = \text{Spec } B, A \subset B \subset K$

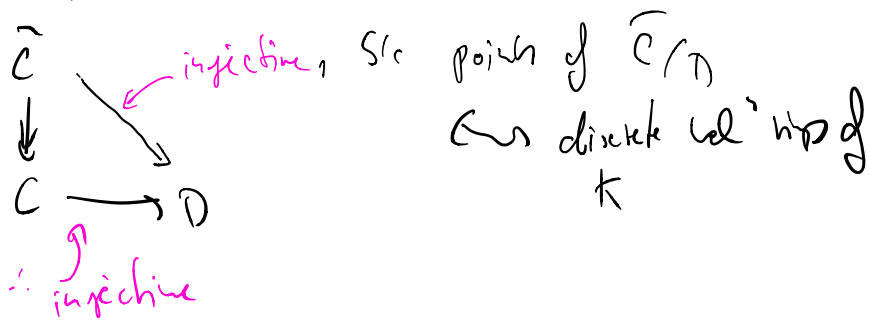
Localize at $x \Rightarrow A$ is a DVR. "common fraction field"

Pick $f \in K \setminus A$. Then $A[f] = K$ s.t. A a DVR.

$\therefore B = K$ or $B = A$.

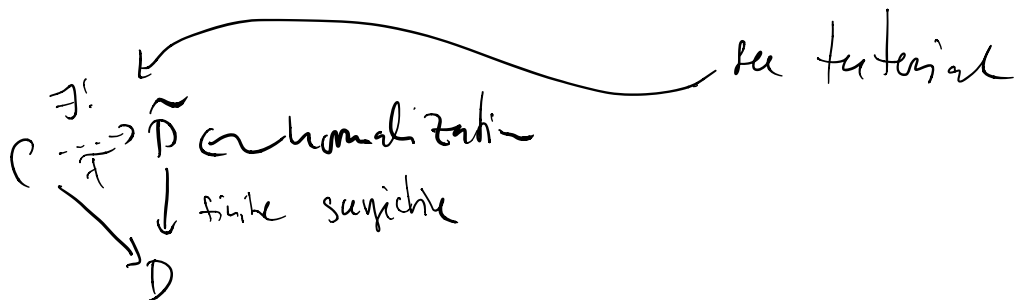
It follows that $C \xrightarrow{f} D$ is a local isom.

To get open immersion, consider universal



Hence $C \rightarrow D$ is an open embedding of top. spaces
 \therefore open immersion. \square

3. $C \xrightarrow{f} D$ dominant morphism of proper curves / k .
 C regular.
 Show: f is finite surj.



C, \tilde{D} proper $\Rightarrow C \xrightarrow{\tilde{f}} \tilde{D}$ proper
 \tilde{D} regular, \tilde{f} flat. $\Rightarrow \tilde{f}$ is open immersion.
 $\therefore \tilde{f}$ is iso

$\therefore f$ finite surj.

4. C proper reg. curve / k X proper k -sch.
 U open dense. $C \xrightarrow{\exists!} X$
 $U \rightarrow X$

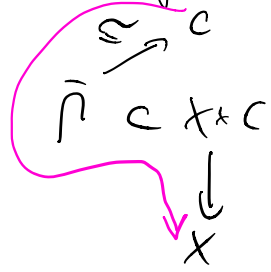
$$X \times C \supset \overline{\Gamma} \quad \Gamma = (y, f(y)) \subseteq U$$

proper curve

Proj¹ $\overline{\Gamma} \rightarrow C$ is biject¹.

By ex 2, $\overline{\Gamma} \rightarrow C$ is open immersion.

But $\bar{\Gamma}, C$ are proper, so $\bar{\Gamma} \xrightarrow{\text{is}} C$.



is the desired lift.

[unique b.c. if $g: C \rightarrow X+C$ is a lift, $g(C) \subset C+C$ is closed & contains $\bar{\Gamma}$, $\therefore g(C) = \bar{\Gamma}$] \square