1. A integral domain of finite type over field k.
Show: A is integrally closed $\iff$ if $\mathfrak{p} \in \operatorname{Spec} A$ has ht 1, then $A_\mathfrak{p}$ is a dvr, & $A = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_\mathfrak{p}$.

$\Leftarrow$: $A_\mathfrak{p}$ dvr $\Rightarrow A_\mathfrak{p}$ i.c. $\Rightarrow \underbrace{\bigcap A_\mathfrak{p}}_{\text{by ass. } A}$ i.c.

$\Rightarrow$: A i.c. $\Rightarrow S^{-1}A$ i.c.

$\left[T^n + \frac{a_1}{s_1} T^{n-1} + \dots + \frac{a_n}{s_n} = 0 \right.$ for some $T \in \operatorname{Frac} A$.

Set $s = \prod_i s_i$, mult^n by s^n $\Rightarrow$ sT is integral over A

$\therefore sT \in A$ $\therefore T \in S^{-1}A \Big]$

$\therefore A_\mathfrak{p}$ i.c. $\forall \mathfrak{p} \in \operatorname{Spec} A$

Need to show: $A = \bigcap_{\operatorname{ht} \mathfrak{p} = 1} A_\mathfrak{p}$.

Use Noether normalization: $\exists A_0 = k[x_1, \dots, x_n] \underset{\text{finite}}{\hookrightarrow} A$.

First prove for A_0, which is a UFD:
- if $\pi \in A_0$ is a prime elt, then (π) is a prime ideal of ht 1. (prime: clear. ht 1: $P \subsetneq (\pi)$, $0 \neq f \in P$, $f = \pi^n \cdot g$, $\pi \nmid g$.

B/c P prime, $g \in P$ or $\pi \in P$) (reduced) $\not\times$ $\not\times$

- $A_{(\pi)}$ = fractions w/ denom. not divisible by π

- $\therefore \bigcap_{\pi} A_{(\pi)}$ = fractions with denominator a unit

$= A$.

For A: $x \in L = \mathrm{Frac}(A)$

$L/K = \mathrm{Frac}(A_0)$ is a finite ext^n.

Note that A is the integral closure of A_0 in L.

$P(T) \in K[T]$ monic min poly of x.

Want to show that P has coeff in A_0. (Then x/A_0 int. $\therefore x \in A$.)

STP P has coeff in $(A_0)_Q$, $Q \in \mathrm{Spec}\, A_0$ has ht 1.

May replace A by A_Q.

Now $\dim A = 1$. $\therefore$ {primes of ht $=1$}

$=$ {max ideals}.

General fact: if A is an i.d., then $A = \bigcap_{m \text{ max.}} A_m$.

This concludes.

Proof of fact: $x \in \mathrm{Frac}(A)$.

$I = \{a \in A \mid x \cdot a \in A\}$

want: $1 \in I$. If $x \in A_m$ $\exists$ $\begin{matrix} a \in A \\ s \in A \setminus m \end{matrix}$ s.t. $x = \frac{a}{s}$

$\therefore s \in I$ $\quad \therefore I \not\subset m$.

$\therefore 1 \in I$ as desired. $\square$

2. $f : C \to D$ $\overset{\text{bir.}}{\vee}$ mor. of curves $/k$ & D is regular.

Show: f is an open imm.

Step 1: for $\underset{\text{closed}}{x \in C}$, $\mathcal{O}_{D, f(x)} \xrightarrow{\cong} \mathcal{O}_{C,x}$.

WMA $D = \operatorname{Spec} A$, $C = \operatorname{Spec} B$, $A \subset B \subset K$ ($=$ comm fraction field.)

Localize at $x \rightsquigarrow A$ is a dvr.

Pick $t \in K \setminus A$. Then $A[t] = K$ s/c A a dvr.

$\therefore B = K$ or $B = A$.
X

It follows that $C \xrightarrow{f} D$ is a local isom^n.

To get open imm, consider normalization

$$\begin{tikzcd} \tilde{C} \arrow[d] \arrow[dr, "\text{injective}"] & \\ C \arrow[r, "\therefore\ \text{injective}"'] & D \end{tikzcd}$$

injective, s/c points of $\tilde{C}/D$ $\leftrightsquigarrow$ discrete val^n rings of K

Hence $C \to D$ is an open embedding of top. spaces

$\therefore$ open imm. $\square$

3. $C \xrightarrow{f} D$ dominant morphism of proper curves $/k$.

C regular.

Show: f is finite surj.

see tutorial

$$\begin{tikzcd} C \arrow[r, dashed, "\exists!"', "\tilde{f}"] \arrow[dr] & \tilde{D} \arrow[d, "\text{finite surjective}"] \\ & D \end{tikzcd}$$

$\tilde{D}$ ← normalization

$C, \tilde{D}$ proper $\Rightarrow$ $C \to \tilde{D}$ proper

$\tilde{D}$ regular, $\tilde{f}$ birat. $\overset{\text{ex 2}}{\Rightarrow}$ $\tilde{f}$ is open imm.

$\therefore \tilde{f}$ is iso

$\therefore f$ finite surj.

4. C proper reg. curve $/k$ $\quad X$ proper k-sch.

$\cup$

U open dense.

$$\begin{tikzcd} C \arrow[dr, dashed, "\exists!"] & \\ U \arrow[u, hook] \arrow[r] & X \end{tikzcd}$$

$X \times C \supset \overline{\Gamma} \qquad \Gamma = (u, f(u)) \cong U$

proper curve

Proj^n $\overline{\Gamma} \longrightarrow C$ is birat^l.

By ex 2, $\overline{\Gamma} \to C$ is open imm.

But $\bar{\Gamma}$, C are proper, so $\bar{\Gamma} \xrightarrow{\simeq} C$.

$$\begin{tikzcd}
 & C \arrow[ddl, bend right=80] \\
\bar{\Gamma} \subset X \times C \arrow[ur, "\simeq"] \arrow[d] & \\
X &
\end{tikzcd}$$

$\nearrow$ is the desired lift.

[unique b/c if $g : C \to X$ is any ext'n, $g(C) \subset C \times X$ is closed & contains Γ $\therefore$ $g(C) = \bar{\Gamma}$] $\square$