1. A UFD

Show: A integrally closed (in its field of frac.)

Pick $\frac{x}{y} \in \mathrm{Frac}(A)$ which is integral: $x, y \in A$

$$\left(\frac{x}{y}\right)^n + a_1\left(\frac{x}{y}\right)^{n-1} + \dots + a_n = 0 \qquad a_i \in A$$

$$\Rightarrow \quad x^n + y\left(a_1 x^{n-1} + a_2 y x^{n-2} + \dots + a_n y^{n-1}\right) = 0$$

$\therefore\ y \mid x^n$

B/c A UFD: every prime factor of y divides x

let y' be one of them.

Replace x by x/y'.

y by y/y' ⟵ one prime factor less than y

Repeating this argument, arrive at $y \in A^\times$

$\therefore\ x/y \in A$.

2. k field

$A = k[T] \qquad B = k[X,Y]/_{Y^3 - X^2}$

(1) Show that $k[T]$ is a Dedekind domain

$$\begin{array}{c} \mathrm{PID} \Rightarrow \mathrm{DD} \\ \psi \\ k[T] \end{array} \qquad \square$$

(2,3) Show that B is not a Dedekind domain & $B \to A$ is an integral closure.

$X \mapsto T^3$
$Y \mapsto T^2$

First approach: the cotangent space of Spec B at (x,y) is

Coker at $3y^2 dy + 2x^2 dx = 0$

At $(0,0)$: no eq^n

$\therefore$ cot. space dim 2

$\therefore$ not regular

Second approach: $\{x^\varepsilon y^n\}$ $\varepsilon \in \{0,1\}$, $n \in \mathbb{N}$

is k-v.s. basis of B.

[LA If R any ring & P is a monic poly of deg d, then $R[T]/P$ is a free R-mod. w/ basis $1, T, \dots, T^{d-1}$.]

Apply to $R = k[Y]$

$k[T]$ has basis $\{T^n\}$ $n \in \mathbb{N}$

$x^\varepsilon y^n \longmapsto T^{3\varepsilon + 2n}$

bij^n between basis & subset T^n $n \geq 1$ of

Hence $A \to k[T]$ is an injection but not iso,

$T \notin$ image

but $T^2 \in$ image.

$T^3 \in$

$\therefore \quad \underbrace{T^3/T^2}_{= T} \in \text{image}(\text{Frac}(A) \to \text{Frac}(k[T]))$

$\therefore \quad \text{Frac}(A) \xrightarrow{\cong} \text{Frac}(k[T])$

To see that A is not integrally closed ($\therefore$ not Ded. dom.)

Note $T \notin A$

$Y = T^2 \in A \qquad \therefore \quad P(T) = 0$

$$P(U) = U^2 - Y$$

$\therefore$ T integral over A but not in A. $\square$