Algebraic Geometry 2

Exercises 4

Dr. Tom Bachmann Summer Semester 2021

Exercise 1. Show that the canonical map $\mathrm{Proj}(A[T_0]) \to \mathrm{Spec}\,(A)$ is an isomorphism.

Exercise 2. Let X be an integral scheme of finite type over the field k. Show that $\dim X = 0$ if and only if $X \simeq \mathrm{Spec}\,(L)$, where L/k is some finite field extension.

Exercise 3. Let A be a commutative ring.

(1) Exhibit a bijection between actions of the group scheme $\mathbb{G}_m$ on $\mathrm{Spec}\,(A)$ and $\mathbb{Z}$-graduations of A.

(2) Given a $\mathbb{G}_m$-action on $\mathrm{Spec}\,(A)$, show that the action extends over the commutative monoid $\mathbb{A}^1 \supset \mathbb{G}_m$ if and only if $A_n = 0$ for $n < 0$.

Exercise 4. Let R be a commutative $\mathbb{N}$-graded ring. Set $A = R_0$ and assume that there are elements $f_0, \ldots, f_r \in R_1$ generating R as an A-algebra.

(1) Put $\mathrm{Spec}\,(R) \setminus \{0\} := \mathrm{Spec}\,(R) \setminus V(R_+)$. Show that $\mathrm{Spec}\,(R) \setminus \{0\}$ is a $\mathbb{G}_m$-invariant open subset of $\mathrm{Spec}\,(R)$ (for the $\mathbb{G}_m$ action coming from the grading via Exercise 3).

(2) Show that $D(f_i) \subset \mathrm{Spec}\,(R) \setminus \{0\}$ is a $\mathbb{G}_m$-invariant open subset which is isomorphic as a scheme with $\mathbb{G}_m$-action to $\mathrm{Spec}\,(R_{(f_i)}) \times \mathbb{G}_m$ (where $\mathbb{G}_m$ acts on the second factor only).

(3) Construct a surjective $\mathbb{G}_m$-equivariant morphism $\mathrm{Spec}\,(R) \setminus \{0\} \to \mathrm{Proj}(R)$ (where the $\mathbb{G}_m$-action on $\mathrm{Proj}(R)$ is trivial).