1. $A[T_0]_+$ is generated as an ideal by T_0.

$\therefore \operatorname{Proj}(A[T_0]) = D_+(T_0)$

$\cong \operatorname{Spec} A[T_0, T_0^{-1}]_0$

$\cong \operatorname{Spec} A$.

2. WMA $\underset{\operatorname{Spec} A}{\overset{\shortparallel}{X}}$ affine (union of spectra of fields is integral $\iff$ only one point)

Noether norm.: $A \underset{\text{finite}}{\supset} k[t_1, \dots, t_n]$. $\operatorname{Dim} A = 0 \implies n = 0$

$\therefore A$ finite$/k$.

For $0 \neq a \in A$, $m_a : A \to A$ is injective (A integral dom.)

$\therefore$ surjective (A f.d. k-v.s.)

$\therefore A$ is a field. $\square$

3. (1) $\mathbb{G}_m$-action on $\operatorname{Spec} A$: $\mathbb{G}_m \times \operatorname{Spec} A \xrightarrow{\beta} \operatorname{Spec} A$

s.t.

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{G}_m \times \operatorname{Spec} A \arrow[r, "m \times id"] \arrow[d, "id \times \beta"'] \arrow[dr, phantom, "G"] & \mathbb{G}_m \times \operatorname{Spec} A \arrow[d] \\
\mathbb{G}_m \times \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A
\end{tikzcd}$$

$$\begin{tikzcd}
A \cong k \times A \arrow[r] \arrow[rrr, bend right, "id"'] & \mathbb{G}_m \times A \arrow[rr] & & A
\end{tikzcd}$$

unit

Dualize: $A[T,T^{-1}] \xleftarrow{\varphi} A$ corr. to β

s.t.

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowright"] & A[T,T^{-1}] \arrow[d, "{\varphi[T,T^{-1}]}"] \\
A[T,T^{-1}] \arrow[r, "T \mapsto TU"'] & A[T,U,T^{-1},U^{-1}]
\end{tikzcd}
\qquad \& \qquad
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[rr, bend right, "id"'] & A[T,T^{-1}] \arrow[r, "T\mapsto 1"] & A
\end{tikzcd}$$

On elements:

$$\begin{tikzcd}
 & a \arrow[r, mapsto] \arrow[dl, mapsto] & \sum_i \varphi_i(a) T^i \arrow[d, mapsto] \\
\sum_i \varphi_i(a) T^i \arrow[d, mapsto] & & \sum_{i,j} \varphi_j(\varphi_i(a)) T^i U^j \\
\sum_i \varphi_i(a) T^i U^i \arrow[urr, phantom, "\overset{!}{=}"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & \sum_i \varphi_i(a) \arrow[d, phantom, "\overset{!}{\shortparallel}"] \\
 & a
\end{tikzcd}$$

$$\therefore \quad \varphi_i \circ \varphi_j = \begin{cases} 0 : & i \neq j \\ \varphi_i : & i = j. \end{cases} \qquad \therefore \quad \sum_i \varphi_i = id$$

Hence φ_i are orthogonal idempotents summing to 1 & hence induce $A \cong \bigoplus_i A_i$.

Conversely, given $A \cong \bigoplus_i A_i$, define $\varphi: A \to A[T,T^{-1}]$

$$a \mapsto \sum_i a_i T^i$$

- Easy to check this is an action.
- Constructions evidently inverse.

(2)
$$\begin{tikzcd}
\mathbb{G}_m \times \operatorname{Spec} A \arrow[r] \arrow[d, hook] & \operatorname{Spec} A \\
\mathbb{A}^1 \times \operatorname{Spec} A \arrow[ur, dashed, "\exists ?"'] &
\end{tikzcd}$$

dualize:
$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr, dashed, "\exists ?"'] & & A[T,T^{-1}] \\
& A[T] \arrow[ur] &
\end{tikzcd}$$

$\varphi(a) = \sum_i \varphi_i(a) T^i$

poss. iff $\varphi_i(a) = 0 \;\forall i<0$, $a \in A$

i.e. $\mathbb{N}$-grading.

4. (1) $\operatorname{Spec} R \setminus \{0\}$ $\mathbb{G}_m$-inv.

$\iff$ 0 $\mathbb{G}_m$-inv.

$\iff$
$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R[T,T^{-1}] \arrow[d, dashed, "\exists ?"] \\
R/I \arrow[r] & R/I[T,T^{-1}]
\end{tikzcd}$$

I = ideal corr. to 0 = $(f_1, \ldots, f_r)$

$\iff$ R/I admits induced grading

$\iff$ I homog.

(2)
$$\begin{tikzcd}
R \arrow[r] \arrow[d] & R[T,T^{-1}] \arrow[d, dashed, "\exists ?"] \\
R_f \arrow[r] & R_f[T,T^{-1}]
\end{tikzcd}$$

$\rightsquigarrow$ Yes as soon as f is homog.

$$
\begin{aligned}
(R_f)_0 \otimes \mathbb{Z}[T, T^{-1}] &\longrightarrow R_f \\
T &\longmapsto f \\
r/f^n &\longmapsto r/f^n
\end{aligned}
$$

is iso of graded rings

$\therefore$ $\qquad \operatorname{Spec} R_f \cong \operatorname{Spec} (R_f)_0 \times \mathbb{G}_m$

as schemes with $\mathbb{G}_m$-action.

(3)
$$
\begin{tikzcd}
D_+(f) \arrow[d, phantom, "\wr\wr"] \arrow[r, phantom, "\subset"] & \operatorname{Proj} R & \\
\operatorname{Spec} R_{(f)} = \operatorname{Spec} (R_f)_0 & & \text{triv. } \mathbb{G}_m\text{-action} \\
\operatorname{Spec} R_f \arrow[u, "\mathbb{G}_m\text{-equiv. by (2)}"'] \arrow[r, phantom, "\subset"] & \operatorname{Spec} R \setminus \{0\} & \text{nat. } \mathbb{G}_m\text{-act.}
\end{tikzcd}
$$

Hence need only show: maps agree on overlaps (by gluing) $\qquad \operatorname{Spec} R_f \cap \operatorname{Spec} R_g$.

But obtain the map corresponding to $f \cdot g$. $\square$