1. $i : X \longrightarrow Y$ open imm.

Show: i proper $\Leftrightarrow$ $Y = X \amalg Z$

$\Rightarrow$: i closed ∴ $i(X) \subset Y$ is an open & closed subset

$Z = Y \setminus i(X)$ — " —

∴ $Y = X \amalg Z$

$\Leftarrow$: properness local on target

Consider cover of Y by X & $Z = Y \setminus X$

$\uparrow$ open by ass.

Base change on $X \xrightarrow{\cong} X$ ← iso. prop.

& $\emptyset \longrightarrow Z$ ← closed imm. ∴ proper

∴ i proper.

2. $f : X \longrightarrow Y$. Show: f sep. $\Leftrightarrow$ $\Delta(X) \subset X \times_Y X$ is closed.

$\Rightarrow$: triv.

$\Leftarrow$: closed imm. is local on target

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X \times_Y X \arrow[d] & \\ & Y & U \arrow[l, hook'] \end{tikzcd}$$

$\rightsquigarrow$ WMA Y is affine

Cover $X = \bigcup_{i \in I} U_i$, $U_i \subset X$ open aff.

$$X \times_Y X = \underbrace{X\times_Y X \setminus \Delta(X)}_{\text{open by ass.}} \cup \underbrace{\bigcup_{i\in I} U_i \times_Y U_i}_{\text{open}}$$

$$\Delta \uparrow \quad \text{is an open cover}$$

X

preimages: $\emptyset$ closed imm. $\qquad U_i$ $\quad \cong U_i$, c.i. s/c U_i & Y are aff.

$\square$

3. $f: X \to Y$ quasi-cpct. mor.

Show: $f(X)$ is closed $\Leftrightarrow$ $f(X)$ is closed under specialization

Aside: $y \in Y$, $\overline{\{y\}} \ni y'$ $\rightsquigarrow$ call y' a <u>specialization</u> of y.

$\Rightarrow$: $\{y\} \subset f(X) \Rightarrow \overline{\{y\}} \subset \overline{f(X)} = f(X)$

$\Rightarrow y' \in f(X)$

$\Leftarrow$: WMA X, Y reduced

Considering $\overline{f(X)}_{\text{red}} \subset Y$, WMA $Y = \overline{f(X)}$.

$$\overline{f(X)}_{\text{red}} \leftarrow X \rightarrow Y$$

Need to prove f surjective. Problem is local on Y.

$\therefore$ WMA Y affine

Then $X = \bigcup_{i=1}^{n} X_i$, $X_i \subset X$ open aff.

Let $y \in Y$. Since $Y = \overline{f(X)} = \bigcup_i \overline{f(X_i)}$

find $y \in \overline{f(X_i)}$ for some i.

Shall find $x \in X_i$ s.t. $y \in \overline{\{f(x)\}}$.

$X_i = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $\alpha : A \to B \leftrightsquigarrow f : X_i \to Y$.

A, B reduced. WMA: $f : X_i \to Y$ dominant.

$\therefore \alpha : A \hookrightarrow B$

$y \in Y \leftrightsquigarrow P \subset A$

Pick $P' \subset P$ minimal prime. (Exists by Zorn's lemma.)

$A_{P'} \hookrightarrow B_{P'}$

field b/c P' minimal

pick a prime ideal Q' (exists b/c $B_{P'} \neq 0$)

$\updownarrow$

prime Q of B ctg P'.

$\therefore$ $Q \leftrightsquigarrow$ pt $x \in X_i$ s.t. $f(x_i)$ specializes to y.

$\therefore$ $y \in f(X)$. $\square$

$\sqcup$

4. S scheme

$X \in Sch_S$ reduced, $Y \in Sch_S$ separated,

$f, g : X \to Y \in Sch_S$ s.t. $f(x) = g(x)$ $\forall x \in U \subset X$ — U open & dense.

Show that $f = g$ as mor. of schemes.

$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \times_Y X \arrow[r, "pr_1"] \arrow[r, shift right=2, "pr_2"'] & X \\
X' \arrow[u] \arrow[r] & X \arrow[u, "\Delta"'] &
\end{tikzcd}
$$

closed imm. onto a subset containing U

$\therefore$ contains $\overline{U} = X$

$\therefore$ $X' = X$ b/c X reduced

$\therefore$ $f = g$.

See also tutorium last Thursday.