

1. $i: X \rightarrow Y$ open imm.

Show: i proper $\Leftrightarrow Y = X \amalg Z$

\Rightarrow : i closed $\therefore i(X) \subset Y$ is an open & closed subset
 $Z = Y \setminus i(X)$ — " —

$$\therefore Y = X \amalg Z$$

\Leftarrow : properness local on target

Consider cover of Y by X & $Z = Y \setminus X$

See charts on $X \xrightarrow{\cong} X \leftarrow \text{loc. prop.}$
 & $\emptyset \rightarrow Z \leftarrow \text{closed imm. } \therefore \text{proper}$
 \uparrow open by ass.

$\therefore i$ proper.

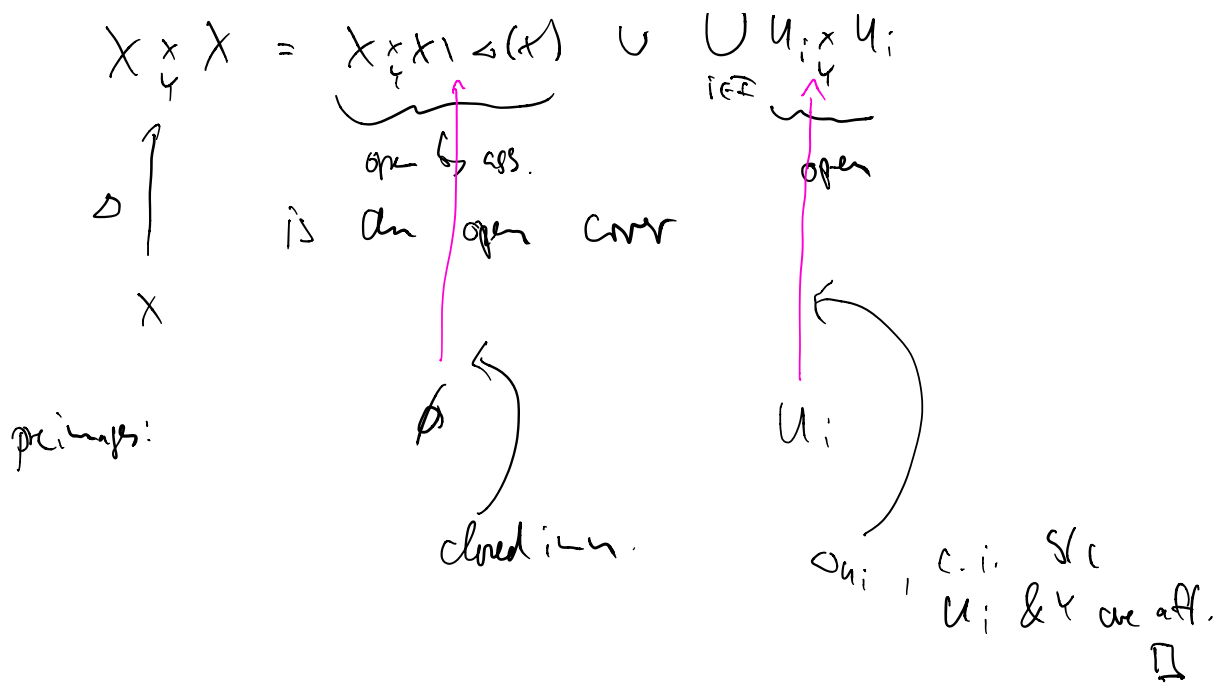
2. $f: X \rightarrow Y$. Show: f sep $\Leftrightarrow \Delta(X) \subset X \times_Y X$ is closed.

\Rightarrow : triv.

\Leftarrow : closed imm. is local on target

$X \rightarrow X \times_Y X$
 $\downarrow \quad \downarrow \quad \leftarrow U \rightsquigarrow$ $\text{w/MA } Y \text{ is affine}$

Cover $X = \bigcup_{i \in I} U_i$, $U_i \subset X$ open aff.



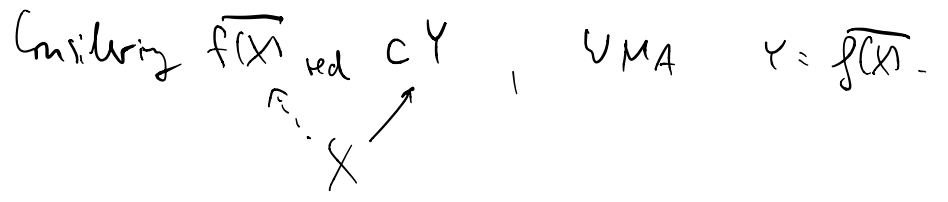
3. $f: X \rightarrow Y$ quasi-coherent morphism.

Show: $f(X)$ is closed $\Leftrightarrow f(X)$ is closed under specialization

Aside: $y \in Y, \overline{\{y\}} \ni y' \rightarrow$ call y' a specialization of y .

\Rightarrow : $\{y\} \subset f(X) \Rightarrow \overline{\{y\}} \subset \overline{f(X)} = f(X)$
 $\Rightarrow y' \in f(X)$

\Leftarrow : WMA X, Y reduced



Need to prove f surjective. Problem is local on Y .

$\therefore \text{WMA } Y \text{ affine}$

Then $X = \bigcup_{i=1}^n X_i$, $X_i \subset X$ open aff.

Let $y \in Y$. Since $Y = \overline{f(X)} = \bigcup_i \overline{f(X_i)}$
hence $y \in \overline{f(X_i)}$ for some i .

Shall find $x \in X_i$ s.t. $y \in \overline{f(x)}$.

$X_i = \text{Spec } B_i$, $Y = \text{Spec } A$, $\alpha: A \rightarrow B \hookrightarrow P: X_i \rightarrow Y$.

A, B reduced. WMA: $f: X_i \rightarrow Y$ dominant.

$\therefore \alpha: A \hookrightarrow B$

$y \in Y \iff P \subset A$

Pick $P' \subset P$ minimal prime. (Exists by Zorn's lemma.)

$A_{P'} \hookrightarrow B_{P'}$
 \uparrow
field $\& P'$ minimal

pick a prime ideal Q'
(exists $\& B_{P'} \neq 0$)

me Q' of B
cts P' .

$\therefore Q \iff \exists x \in X_i$ s.t. $f(x)$ specializes to y .

$\therefore y \in f(X)$. \square

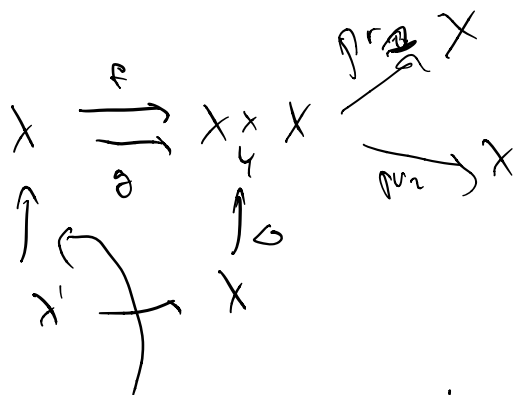
□

4. S scheme

$X \in \text{Schemes}$ reduced, $Y \in \text{Schemes}$ separated,

$f, g: X \rightarrow Y \in \text{Schemes}$ s.t. $f(x) = g(x) \forall x \in U \subset X$

Show that $f = g$ as mor. of schemes. U open & dense.



closed immersion into a subset containing U

$$\therefore \text{closure } \bar{U} = X$$

$$\therefore X' = X \text{ s.t. } X \text{ reduced}$$

$$\therefore f = g.$$

See also tutorial on last Thursday.