1. X a scheme, $j: U \hookrightarrow X$ open immersion.

(1) Show that $j^*(F) \simeq j^{-1}(F) = F|_U$

$\forall$ $\mathcal{O}_X$-mod F.

$$j^*(F) = j^{-1}(F) \otimes_{j^{-1}(\mathcal{O}_X)} j^{-1}(\mathcal{O}_U).$$

$\therefore$ WMA $F = \mathcal{O}_X$, i.e. show that $j^{-1}(\mathcal{O}_X) \cong \mathcal{O}_U$.

True ess. by def^n.

(1') Deduce that j^* is exact.

Clear sc $(\ )|_U$ is exact, since for $x \in U$, $(F|_U)_x \cong F_x$.

(2) Construct closed imm. $i: Z \hookrightarrow X$ s.t. i^* is not exact.

Take e.g. $X = \mathbb{A}^1_k$, $Z = \{0\} \hookrightarrow \mathbb{A}^1_k$
$\cong \operatorname{Spec} k$.

$$\begin{array}{ccc} i^*: \mathrm{QCoh}(\mathbb{A}^1_k) & \longrightarrow & \mathrm{QCoh}(Z) \\ \| & & \| \\ k[t]\text{-mod} & & k\text{-mod} \\ M & \longmapsto & M/tM \end{array}$$

$$\left(k[t] \xrightarrow{\cdot t} k[t]\right) \longmapsto \left(k \xrightarrow{\cdot 0} k\right)$$

injective $\qquad$ not injective

2. $X \in \mathrm{Top}$

(1) $0 \to F_1 \to F_2 \to F_3 \to 0$, F_1 flasque

$\Rightarrow 0 \to F_1(X) \to F_2(X) \to F_3(X) \to 0.$

"proof": $0 \to H^0(X, F_1) \longrightarrow H^0(X, F_2) \to H^0(X, F_3) \to \underbrace{H^1(X, F_1)}_{=0 \text{ sc } F_1 \text{ is flasque.}}$

proof by hand: $\Gamma(X, -)$ is left exact

$\therefore$ need only prove that $F_2(X) \twoheadrightarrow F_3(X)$.

Let $\alpha \in F_3(X)$. Consider the set $\{(V, \alpha') \mid V \subset X$ open, $\alpha' \in F_2(V)$ lifts $\alpha|_V\}$. Partially ordered by inclusion/refinement.

- non-empty
- chains have upper bounds by

sheaf property

Zorn's lemma $\rightsquigarrow$ max. elt (V, α').

STP $V = X$. If not, $\exists\, x \in X \setminus V$.

Since $F_2 \twoheadrightarrow F_3$, $\exists$ open nbd. U of x & $\alpha_1 \in F_2(U)$

lifting α. $\quad \alpha' - \alpha_1 \in F_1(U \cap V)$

$$\exists \ \delta \in F_1(X) \text{ st. } \delta|_{U \cap V} = \alpha_1$$

$$\alpha_2 = \alpha_1 + \delta \in F_2(U)$$

Now
$$\begin{aligned} \alpha' - \alpha_2 &= \alpha' - (\alpha_1 + \delta) \\ &= (\alpha' - \alpha_1) - \delta \\ &= 0 \end{aligned}$$

$\therefore$ α', α_2 determine lift of α over $U \cup V$.

(2) $0 \to F_1 \to F_2 \to F_3 \to 0$, F_1 & F_3 flasque.

$\Longrightarrow$ F_2 flasque.

$U \subset V \subset X$

$$\begin{array}{ccccccccc} 0 & \to & F_1(V) & \to & F_2(V) & \longrightarrow & F_3(V) & \xrightarrow{(1)} & 0 \\ & & \big| & & \big| & & \big| & & \big| \end{array}$$

$$0 \to F_1(U) \to F_2(U) \to F_3(U) \to 0$$

surj. by 5-lemma

(3) If $f : X \to Y$ is cts & F is flasque on X then $f_* F$ is flasque on Y.

Let $U \subset V \subset Y$.

$$\begin{tikzcd}
(f_*F)(V) \arrow[r, phantom, "="] \arrow[d, "\text{surj?}"'] & F(f^{-1}V) \arrow[d, "\text{surj. by ass.}"] \\
(f_*F)(U) \arrow[r, phantom, "="] & F(f^{-1}(U))
\end{tikzcd}$$

3. $\mathcal{E}$ loc. free rk n on X.

$$\begin{tikzcd}
V(\mathcal{E}) = \underline{\mathrm{Spec}}(\mathrm{Sym}\,\mathcal{E}) \arrow[d, "p"] \\
X
\end{tikzcd}$$

(1) Show that $V(\mathcal{E}) \cong \mathbb{A}^n_X$ locally on X.

Local on X $\rightsquigarrow$ WMA $\mathcal{E} \cong \mathcal{O}^n$

WMA $U = \mathrm{Spec}\, A$

$$\mathcal{E} \hookrightarrow A^n$$

$$\mathrm{Sym}\,\mathcal{E} \hookrightarrow A[T_1, \dots, T_n]$$

$$\mathrm{Spec}\, A[T_1, \dots, T_n] = \mathbb{A}^n_A \text{ as } A\text{-sch.}$$

(2) $\underbrace{\Gamma(-, V(\mathcal{E}))}_{\substack{\| \\ U \longmapsto \left\{ \begin{smallmatrix} & & V(\mathcal{E}) \\ & \sigma \nearrow & \downarrow p \\ U & \hookrightarrow & X \end{smallmatrix} \right\}}} \cong \mathcal{E}^\vee.$

$$(\mathcal{E}^\vee)(U) = \{\sigma : \mathcal{E}|_U \to \mathcal{O}_U\}$$

σ yields an augmentation $\mathrm{Sym}_U\, \mathcal{E}|_U \to \mathcal{O}_U$

& hence $\underline{\mathrm{Spec}}\, \mathcal{O}_U \longrightarrow \underline{\mathrm{Spec}}\, \mathrm{Sym}\, \mathcal{E}|_U$

$$\begin{array}{ccc} \| & & \| \\ U & & V(\mathcal{E})|_U. \end{array}$$

Hence have built $\mathcal{E}^\vee \longrightarrow \Gamma(-, V(\mathcal{E}))$.

To prove iso, may work locally.

Hence WMA $\mathcal{E} \cong \mathcal{O}^n$, $V(\mathcal{E}) = \mathbb{A}^n_U$

$U = \mathrm{Spec}\, A.$

$$\mathcal{E}^\vee(U) = \mathrm{Hom}_{A\text{-mod}}(A^n, A)$$

$\cong$ ← iso via the canonical map

$$\Gamma(-, \mathbb{A}^n_n) = \mathrm{Hom}_{A\text{-alg}}(A[T_1, \dots, T_n], A)$$

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4. $I \in Ab(X)$ injective

$U \overset{j}{\hookrightarrow} X$ open. Show: $I|_U$ is injective.

$$\begin{tikzcd} F \arrow[r] \arrow[d, hook] & j^{-1} I \\ G \arrow[ur, dashed, "e"'] & \end{tikzcd}$$

Suppose that j^{-1} has an exact left adjoint $j_!$. Then e exists $\Leftrightarrow$

$$\begin{tikzcd} j_! F \arrow[r] \arrow[d, hook, "\text{inj. by ass.}"'] & I \\ j_! G \arrow[ur, dashed, "e^+"'] & \end{tikzcd}$$

e^+ exists.

But I is injective, so e^+ always exists.

[Any functor with an exact left adjoint preserves injectives.]

$j_!^p : (\text{Abelian presh. on } U) \longrightarrow (\text{Ab presh. on } X)$

$$(j_!^p F)(V) = \begin{cases} F(V) & \text{if } V \subset U \\ 0 & \end{cases}$$

Then easy to see that $j_!^p \dashv j^{-p}$ for presheaves.

Now if F a sheaf on U, G on X then

$$\begin{aligned} \mathrm{Hom}_{Sh}(F, j^{-1}G) &= \mathrm{Hom}_{PSh}(F, j^{-1}G) \\ &= \mathrm{Hom}_{PSh}(F, j^{-p}G) \\ &= \mathrm{Hom}_{PSh}(j_!^p F, G) \\ &= \mathrm{Hom}_{Sh}(a j_!^p F, G) \end{aligned}$$

$\therefore$ $j_! := a j_!^p \dashv j^{-1}$.

$$(j_! F)_x = \begin{cases} F_x & : x \in U \\ 0 & : x \notin U. \end{cases}$$

$\therefore$ $j_!$ is exact. $\square$

Deduce $R^i f_* F$ is the sheaf ass. with the presheaf $V \longmapsto H^i(f^{-1}V, F)$.

By defn, $R^i f_* F \in \mathrm{Shv}(Y)$ is computed by taking injective res' $F \to I^0 \to I^1 \to I^2 \to \cdots$

& then $R^i f_* F = h^i\left(f_* I^0 \to f_* I^1 \to \cdots \right)$

$$= a \, \frac{\ker(f_* I^i \to f_* I^{i+1})}{\operatorname{im}(f_* I^{i-1} \to f_* I^i)}$$ ← as presh.

→ Sections on V: $\dfrac{\ker(I^i(U) \to I^{i+1}(U))}{\operatorname{im}(I^{i-1}(U) \to I^i(V))}$

$$\cong H^i(V, F|_V)$$

s/c $F|_V \to I^0|_U \to I^1|_V \to \cdots$

is an injective resn by 1st part. $\square$