1. Define

$$B \underset{A}{\otimes} N^{\varphi} \xrightarrow{c_N} N \qquad M \xrightarrow{u_M} (B \underset{A}{\otimes} M)^{\varphi}$$

$$b \otimes n \longmapsto bn \qquad m \longmapsto 1 \otimes m.$$

Clearly nat^{l} transformations. Check these are unit & co-unit of an adjunction:

$$B \underset{A}{\otimes} M \xrightarrow{B \otimes u_M} B \underset{A}{\otimes} B \underset{A}{\otimes} M \xrightarrow{c_{B \otimes M}} B \underset{A}{\otimes} M$$

$$b \otimes m \longmapsto b \otimes 1 \otimes m \longmapsto b \cdot (1 \otimes m) = b \otimes m$$

is the identity.

$$N \xrightarrow{u_{N^{\varphi}}} B \underset{A}{\otimes} N \xrightarrow{(c_N)^{\varphi}} N$$

$$n \longmapsto 1 \otimes n \longmapsto n$$

is also the identity. $\square$

2. $(f_* \tilde{N})(D_A(g)) = \tilde{N}(D_B(\varphi g)) = N \underset{B}{\otimes} B[1/\varphi g]$

$\cong N^{\varphi} \underset{A}{\otimes} A[1/g]$

$= \widetilde{N^{\varphi}}(D_A(g))$.

These isos are compatible as g varies & hence induce an isomorphism of sheaves.

3. Tricky!

RTP: Given A-module Σ & $(f_1, \dots, f_n) = 1$ s.t. $\Sigma_{f_i} \cong A_{f_i}$, then $\Sigma \cong A$.

Note: $\mathrm{Hom}(\Sigma, A)_{f_i} \cong (\Sigma_{f_i})^\vee \cong A_{f_i}$.

$\therefore \exists\, 0 \neq \alpha \in \mathrm{Hom}(\Sigma, A)$.

Then $\alpha : \Sigma \longrightarrow A$ is injective as may be checked locally.

$\therefore$ WMA $\underset{\substack{\| \\ I}}{\Sigma} \hookrightarrow A$ is an ideal.

Pick $0 \neq f \in I$ & write $f = \overset{\text{unit}}{u} \cdot \prod_{i=1}^{m} \overset{\text{prime}}{p_i^{e_i}}$

Since $I_{(p_i)} \cong A_{(p_i)}$ [$A_{(p_i)}$ is local!]

$\exists\, a_i \in A \setminus (p_i)$ s.t. $I_{a_i} \cong A_{a_i}$, say $I_{a_i} = (g_i)$.

Write $g_i = p_i^{c_i} \cdot g_i'$ where $p_i \nmid g_i'$ (NB: loc^n of UFD is UFD. Also p_i not a unit in A_{a_i} by constr^n.)

Have $c_i \leq e_i$.

Claim: $\left(\prod_{i=1}^{m} p_i^{c_i}\right) = I$.

May check locally, so suppose $I_q = (g)$.
Let $J \subset \{1 \leq j \leq n \mid p_j$ not a unit in $A_q\}$.
Since $g \mid f$ we have $g = \overset{\text{unit}}{v} \prod_{j \in J} p_j^{d_j}$.

Since $\underset{\underset{(g)}{\parallel}}{I_{qa_j}} = \underset{\underset{(g_J)}{\parallel}}{I_{a_j q}}$ we must have $d_j = c_j$.

This concludes the proof. $\square$

4. (1) By const^n, for any $\mathcal{O}_X$-module M,
$(M^\vee)|_U \cong (M|_U)^\vee$.

Hence to check$_n$ that $\Sigma \longrightarrow (\Sigma^\vee)^\vee$ is iso,
WMA $\Sigma \cong \mathcal{O}_X$. This case is clear.

(2) $\Sigma^\vee \otimes M \longrightarrow \underline{H}(\Sigma, M)$

$\xleftrightarrow{\text{adj.}}$ $\Sigma \otimes \Sigma^\vee \otimes M \longrightarrow M$

take $ev_\Sigma \otimes id_M$, $ev_\Sigma : \Sigma \otimes \Sigma^\vee \longrightarrow \mathcal{O}$
the co-unit.

(More succinctly: $\Sigma^\vee \otimes M \to \underline{H}(\Sigma, M)$

$$f \otimes m \longmapsto (\sigma \mapsto f(\sigma) \cdot m)$$

May check iso locally, so WMA $\Sigma = \mathcal{O}^n$.
In this case both sides are $\cong M^n$, & check immediately that map is canonical iso. $\square$