Algebraic Geometry 2

Exercises 10

Dr. Tom Bachmann Summer Semester 2021

Exercise 1. Let X be a scheme and $U \subset X$ an open subset.

(1) Let $\mathbb{Z}_U$ be the sheaf on X associated with the presheaf
$$V \mapsto \begin{cases} \mathbb{Z} & V \subset U \\ 0 & V \not\subset U \end{cases}.$$
Show that for any abelian sheaf F we have
$$\mathrm{Hom}_{Ab(X)}(\mathbb{Z}_U, F) \simeq F(U).$$

(2) Let $\underline{\mathcal{O}}_U = \mathcal{O}_X \otimes \mathbb{Z}_U$. Show that for any sheaf of $\mathcal{O}_X$-modules M we have
$$\mathrm{Hom}_{\mathcal{O}_X}(\underline{\mathcal{O}}_U, M) \simeq M(U).$$

(3) Show that $\underline{\mathcal{O}}_U$ is *not* quasi-coherent in general.

Exercise 2. Let X be a noetherian scheme, M a coherent $\mathcal{O}_X$-module and $x \in X$. Show that if $M_x \simeq \mathcal{O}_{X,x}$ (as $\mathcal{O}_{X,x}$-modules) then there exists an open neighbourhood U of x in X such that $M|_U \simeq \mathcal{O}_X|_U$.

Exercise 3. Let X be a scheme and $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules.

(1) Show that the presheaf
$$\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F}) : U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U, \mathcal{F}|_U)$$
is a sheaf of $\mathcal{O}_X$-modules.

(2) Show that there is a natural isomorphism
$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}).$$