1. (1) Let $\mathbb{Z}_U^p = \left(V \longmapsto \begin{cases} \mathbb{Z} & V \subset U \\ 0 & \text{else} \end{cases}\right)$. STP

$$\mathrm{Hom}_{\mathrm{AbPSh}}(\mathbb{Z}_U^p, F) \cong F(U) \quad \forall \text{ presh. } F.$$

Given $\alpha : \mathbb{Z}_U^p \longrightarrow F$, obtain $c := \alpha(U)(1) \in F(U)$.

Let $V \subset U$. Then

$$\begin{tikzcd} \mathbb{Z}_U^p(U) \arrow[r, "\alpha(U)"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & F(U) \arrow[d] \\ \mathbb{Z}_U^p(V) \arrow[r, "\alpha(V)"] & F(V) \end{tikzcd}.$$

If $V \not\subset U$ then $\underset{\emptyset}{\underset{\parallel}{\mathbb{Z}_U^p(V)}} \xrightarrow[\exists!]{\alpha(V)} F(V)$.

It follows that c determines $\alpha(V)$ $\forall V \subset X$, & hence all of α.

Conversely, given any $c \in F(U)$, build α by above prescription ($\mathbb{Z} = \mathbb{Z}_U^p(U) \longrightarrow F(U)$ def'd by $1 \mapsto c$). //

(2) STP for presheaves of $\mathcal{O}_X$-modules.

$$\mathrm{Hom}_{\mathcal{O}_X}(\underline{\mathcal{O}}_U^p, F) = \left\{ \text{compat. fam. of elts } \mathrm{Hom}_{\mathcal{O}_X(V)}(\underline{\mathcal{O}}_U^p(V), F(V)) \right\}$$

But $\underline{\mathcal{O}}_U^p(V) = \underline{\mathbb{Z}}_U^p(V) \otimes \mathcal{O}_X(V)$, so $\mathrm{Hom}_{\mathcal{O}_X(V)}(\underline{\mathcal{O}}_U^p(V), F(V))$

$$\cong \mathrm{Hom}_{Ab}(\underline{\mathbb{Z}}_U^p(V), F(V))$$

$\therefore \quad \cong \{ \text{compat. fam. } \mathrm{Hom}_{Ab}(\underline{\mathbb{Z}}_U^p(V), F(V))$

$\cong \mathrm{Hom}_{AShPk}(\underline{\mathbb{Z}}_U^p, F)$

$\cong F(U)$ by (1).

RMK: Adjunction $AbSh \underset{\text{forget}}{\overset{\otimes \mathcal{O}_X}{\rightleftarrows}} \mathcal{O}_X\text{-mod}$ ...

(3) Take $X = \mathrm{Spec}\, A$, A a local ring $(\dim > 0)$, $\emptyset \neq U \subsetneq X$.

Then $\underline{\mathbb{Z}}_U(X) = \underline{\mathbb{Z}}_U^p(X) = 0$.

But $\underline{\mathbb{Z}}_U \neq 0$ (else $F(U) = 0$ $\forall$ sheaves F – false for $\mathcal{O}_X$)

$\therefore$ not q.coh. //

2. WMA $X = \mathrm{Spec}\, A$, A noeth.

$x = \mathfrak{p}$, M a f.g. A-mod.

Pick generator $\frac{m}{a}$ $(a \in A \setminus \mathfrak{p})$ of $M_\mathfrak{p} \cong A_\mathfrak{p}$. Then m also generates.

Obtain map $A \xrightarrow{\varphi} M$.
$1 \longmapsto m$

<u>Lemma</u> If A a ring, N a f.g. A-module, $\mathfrak{p} \in \mathrm{Spec}\, A$,

$N_\mathfrak{p} = 0$. Then $N_f = 0$ for some $f \in A \setminus \mathfrak{p}$.

Applying this to $\ker \varphi$ & $\mathrm{coker}\, \varphi$, obtain f_1, f_2. Put $U = D(f_1 \cdot f_2)$. $\square$

Pf of lemma Given gens $n_1, \dots, n_r \in N$, their imgs in N_p vanish, so $n_i \cdot f_i = 0$ for some $f_i \in A \setminus p$.
Take $f = \prod_i f_i$: $f n_i = 0$ $\forall i$, so $fN = 0$, so $N_f = 0$. $\square$

3. (a) Must show that morphisms of sheaves glue uniquely.

Suppose given cover U_α of X.

- Given $f_1, f_2 : \mathcal{E} \to \mathcal{F}$ s.t. $f_1|_{U_\alpha} = f_2|_{U_\alpha}$.

  Then $\forall V \subset X$, $a \in \mathcal{E}(V)$, $f_1(a)|_{U_\alpha \cap V} = f_2(a)|_{U_\alpha \cap V}$

  $\therefore f_1(a) = f_2(a)$

  $\therefore f_1 = f_2$.

- Given $f_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha}$ s.t. $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$

  Let $V \subset X$, $a \in \mathcal{E}(V)$. Then the $f_\alpha(a|_{V \cap U_\alpha})$ form a compat. family & so glue uniquely to $f_\alpha(a) \in \mathcal{F}(V)$. //

(2) Define $\underline{\mathrm{Hom}}(F, G) \otimes F \xrightarrow{c} G$

as sheafification of $f \otimes a \longmapsto f(a)$.

Define $\mathcal{E} \xrightarrow{u} \underline{\mathrm{Hom}}(F, \mathcal{E} \otimes F)$

as $e \longmapsto (f \mapsto e \otimes f)$.

Can form

$$\begin{tikzcd}
 & \mathrm{Hom}(E, \underline{\mathrm{Hom}}(F,G)) \arrow[dl, "\otimes F"', "a"] & \\
\mathrm{Hom}(E\otimes F, \underline{\mathrm{Hom}}(F,G)\otimes F) \arrow[dr, "c_*", "b"'] & & \mathrm{Hom}(\underline{\mathrm{Hom}}(F,E\otimes F), \underline{\mathrm{Hom}}(F,G)) \arrow[ul, "u^*"', "d"] \\
 & \mathrm{Hom}(E\otimes F, G) \arrow[ur, "\underline{\mathrm{Hom}}(F,-)"', "c"] &
\end{tikzcd}$$

& need to show that ζ $\jmath$ are inverse bijections.

$\jmath \circ \zeta$: Suppose given $E \xrightarrow{\alpha} \underline{\mathrm{Hom}}(F,G)$.

$\rightsquigarrow a, b$

$$E\otimes F \xrightarrow{\alpha\otimes F} \underline{\mathrm{Hom}}(F,G)\otimes F \xrightarrow{c} G$$

$\rightsquigarrow c, d$

$$E \xrightarrow{u} \underline{\mathrm{Hom}}(F, E\otimes F) \xrightarrow{\underline{\mathrm{Hom}}(F,\alpha\otimes F)} \underline{\mathrm{Hom}}(F, \underline{\mathrm{Hom}}(F,G)\otimes F) \xrightarrow{\underline{\mathrm{Hom}}(F,c)} \underline{\mathrm{Hom}}(F,G)$$

On sections:

$$e \longmapsto (f \mapsto e\otimes f) \longmapsto (f \mapsto \alpha(e)\otimes f) \longmapsto \underbrace{(f \mapsto \alpha(e)(f))}_{=\alpha(e)}$$

In other words $\jmath \circ \zeta = id$.

$\zeta \circ \jmath$: Proved similarly.

//