1. $X \to Y$, $X' \to Y'$ $\in \mathrm{Sch}_S$ closed imm.

Show: $X \times_S X' \longrightarrow Y \times_S Y'$ is c.i.

Affine case: ✓ $S = \operatorname{Spec} A$ $Y = \operatorname{Spec} B$ $X = \operatorname{Spec} C$

$Y' = \operatorname{Spec} B'$ $X' = \operatorname{Spec} C'$

$B \twoheadrightarrow C$

$B' \twoheadrightarrow C'$

$B \otimes_A B' \longrightarrow C \otimes_A C'$

is still surjective. ✓

LA Let $f : X \to Y \in \mathrm{Sch}$, $\{U_i\} \subset Y$ open cover.

Then f is a c.i. $\Leftrightarrow$ $f_i : X \times_Y U_i \to U_i$ c.i. $\forall i$.

Pf Recall - f c.i. $\Leftrightarrow$ f affine & $f^\# : \mathcal{O}_Y \twoheadrightarrow f_* \mathcal{O}_X$

- f affine $\Leftrightarrow$ f_i affine

$\therefore$ WMA f affine, RTP $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surj.

$\Leftrightarrow$ $f_i^\#$ is $\forall i$. Clear b/c sheaves are local. $\square$

Let $S_0 \subset S$ open aff.

$Y_0 \subset Y \times_S S_0$ open aff

$Y_0' \subset Y' \times_S S_0$ — " —

- Y is covered by open subsets of form Y_0

$X_0 = X \times_Y Y_0$

$X_0' = X' \times_{Y'} Y_0'$.

– Y' is ---

– $Y \times_S Y'$ is covered by opens of the form $Y_0 \times_{S_0} Y_0'$

$$\therefore \text{ RTP } \overbrace{(X \times_S X') \times_{Y \times_S Y'} (Y_0 \times_{S_0} Y_0')}^{X_0' \times_{S_0} X_0 \ =} \longrightarrow Y_0 \times_{S_0} Y_0$$

is a c.i.

$\therefore$ Reduced problem to Y, Y', S all affine.

$\rightsquigarrow X, X'$ affine (b/c $X \to Y$ c.i.)

$\rightsquigarrow$ already dealt with. $\square$

2. $\underbrace{X \xrightarrow{f} Y \xrightarrow[\text{sep.}]{g} S}_{\text{aff}}$. Show: f is affine.

check using functor of points

$$\begin{tikzcd}
X \arrow[r, hook, "\therefore \text{aff}"] \arrow[d] \arrow[rr, bend left, "f"] & X \times_S Y \arrow[r, "\therefore \text{aff}"] \arrow[d] \arrow[dr] & Y \arrow[d] \\
Y \arrow[r] & Y \times_S Y & X \arrow[r] & S
\end{tikzcd}$$

aff. aff

4. $f : X \to Y$. f is sep. $\iff \exists$ open covering $\{U_i\} \subset Y$ s.t. $f_i : X \times_Y U_i \to U_i$ is sep.

I.e.

$$\begin{tikzcd}
X \arrow[d, "\text{is c.i.}"'] & X \times_Y U_i = X_i \arrow[l] \arrow[d, "\text{iff. all } a.i."] \\
X \times_Y X \arrow[d] & X_i \times_{U_i} X_i \arrow[l] \arrow[d] \\
Y & U_i \arrow[l]
\end{tikzcd}$$

This is exactly the L.A. from Q1.

3. R val^n ring, K field of frac. of R.

X scheme

$T = \operatorname{Spec} R$. $y_0 \in T$ closed pt

$y_1 \in T$ gen. pt

Show: $\operatorname{Hom}_{Sch}(T, X) \xrightarrow{F} \left\{ (x_0, x_1, \alpha) \;\middle|\; \begin{array}{l} x_1 \in X \\ x_0 \in \overline{\{x_1\}} = Z \\ \alpha : k(x_1) \to K \\ \text{s.t. } R \text{ dominates} \\ \quad \mathcal{O}_{Z, x_0} \end{array} \right\}$

Suppose given $T \to X$. $x_0 \in \overline{\{x_1\}}$?

Note: $x_0 \in \overline{\{x_1\}}$

$y_0 \longmapsto x_0$

$y_1 \longmapsto x_1$

$k(x_1) \xrightarrow{\alpha} k(y_1) = K.$

LA cts. maps pres. specialization.

Pf $f: X \to Y$ cts. map. $x_1 \in X$, $x_0 \in \overline{\{x_1\}}$. $y_i := f(x_i)$.

Let $Z \subset Y$ closed, $y_1 \in Z$.

Need: $y_0 \in Z$. $f^{-1}(Z) \subset X$ closed

$x_1 \in f^{-1}(Z)$

$\therefore x_0 \in f^{-1}(Z)$

$\therefore y_0 \in Z$ $\square$

Does R dominate $\mathcal{O}_{Z,x_0}$?

I.e.

$$\begin{tikzcd} R \arrow[r, "\text{local hom}"] \arrow[d, hook] & \mathcal{O}_{Z,x_0} \arrow[d] \\ K \arrow[r] & \mathcal{O}_{Z,x_1} \end{tikzcd}$$

i.e. induced map on stalks

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & X \\ & Z \arrow[u, hook] \end{tikzcd}$$

General fact: Given $X \xrightarrow{f} Y$ mor. of schemes, X reduced

get $X \to \overline{f(X)} \hookrightarrow Y$.

Why? Affine case: Given

$$\begin{tikzcd} A & B \arrow[l, "\alpha"'] \arrow[dl] \\ \alpha(B) \arrow[u, hook] & \end{tikzcd}$$

A reduced, $\alpha(B)$ reduced

I.e. produce factorization through closed immersion.

Exercise: $\operatorname{Spec} \alpha(B) = f(\operatorname{Spec} A)$.

$\therefore$ well-defined map

<u>injectivity</u> $f_1, f_2 : T \longrightarrow X$

$F(f_1) = F(f_2)$

$f_1(y_i) = f_2(y_i)$. $\qquad f_i(y_0) = x_0$

The f_i factor through $\operatorname{Spec} \mathcal{O}_{X, x_0}$ uniquely.

$\therefore$ WMA $X = \operatorname{Spec} A$ $\quad$ A local ring

$f_i(x_0) =$ closed pt of A

$f_i(x_1) = P$ some prime ideal.

$$\begin{tikzcd}
R \arrow[r, hook] & K \\
A \arrow[u, shift left, "f_2^\#"'] \arrow[u, shift right, "f_1^\#"] \arrow[r] & k(P) \arrow[u, "\alpha"']
\end{tikzcd}$$

$\therefore f_1^\# = f_2^\#$

$\therefore f_1 = f_2$.

<u>surjectivity</u> Suppose given $x_1 \in X$

$x_0 \in \overline{\{x_1\}}$

$\alpha : k(x_0) \to K$

R dominates $\mathcal{O}_{Z, x_0}$.

Need to find $f: T \to X$ s.t. $F(f) = (x_0, x_2, \alpha)$.

Find.

$$\begin{tikzcd} T \arrow[r, dashed, "?"] & Z \arrow[r, hook] & X \end{tikzcd}$$

$$\begin{tikzcd} K & k(x_0) \arrow[l, "\alpha"'] \\ R \arrow[u, hook] & \mathcal{O}_{Z, x_0} \arrow[u, hook] \arrow[l, dashed, "?"] \end{tikzcd}$$

exists b/c domination & is a local hom.

This does the job. $\square$