

1. $X \rightarrow Y \in \text{Sch}_S$ closed imm.

$X' \rightarrow Y'$ Show: $X \times_S X' \rightarrow Y \times_S Y'$ is c.i.

Affine case: $S = \text{Spec } A$ $Y = \text{Spec } B$ $X = \text{Spec } C$

$Y' = \text{Spec } B'$ $X' = \text{Spec } C'$

$$B \otimes_A B' \rightarrow C \otimes_A C'$$

$$B \rightarrow C$$

$$B' \rightarrow C'$$

is still surjective. /

LA Let $f: X \rightarrow Y \in \text{Sch}$, $\{U_i\} \subset Y$ open cover.

Then f is a c.i. $\Leftrightarrow f_i: X \times_Y U_i \rightarrow U_i$ c.i. $\forall i$.

Pf Recall - f.c.i. $\Leftrightarrow f$ affine & $f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$

- f affine $\Leftrightarrow f_i$ affine

\therefore wma f affine, RTP $f^\#: \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is surj.

$\Leftrightarrow f_i^\#$ is $\forall i$. clear $\forall i$ means
an local. \square

Let $S_0 \subset S$ open aff.

$Y_0 \subset Y \times_S S_0$ open aff

$Y'_0 \subset Y' \times_S S_0$ - c.i. -

- Y is covered by
open subsets of
form U_0

$$x_0 = X \times Y_0$$

$$x_0' = X' \times Y_0'$$

- Y' is ---

- $Y \times Y'$ is covered by
ops of the R

$$Y_0 \times Y_0'$$

$$\therefore \text{PTP } \underbrace{(X \times X') \times (Y_0 \times Y_0')}_{S_0} \longrightarrow Y_0 \times Y_0'$$

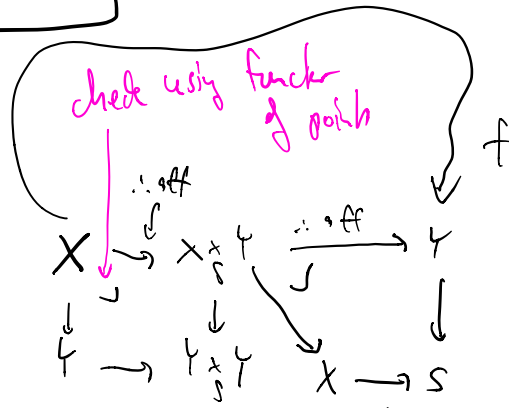
is a c.i.

\therefore Reduced probn to Y, Y', S all effn.

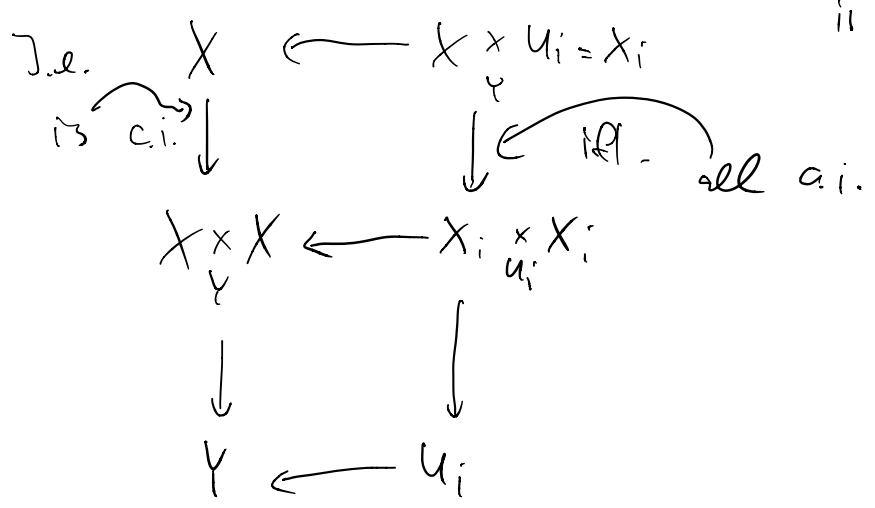
$\rightsquigarrow X, X'$ effn (bc $X \rightarrow Y$ c.i.)

\rightsquigarrow already dealt with. \square

2. $X \xrightarrow{f} Y \xrightarrow[\text{Sep.}]{g} S$. Show: f is effn.



4. $f: X \rightarrow Y$. f is sep. $\Leftrightarrow \exists$ ^{open} covering $\{U_i\} \subset Y$
 s.t. $f_i: X \times_Y U_i \rightarrow U_i$
 is sep.



This is exactly the L.A. for Q1.

3. R valⁿ ring, K field of frac. of R .
 X scheme
 $T = \text{Spec } R$.

$y_0 \in T$ closed pt
 $y_2 \in T$ gen. pt

Show: $\text{Hom}_{\text{Set}}(T, X) = \left\{ (x_0, x_2, \alpha) \mid \begin{array}{l} x_2 \in X \\ x_0 \in \overline{\{x_2\}} = Z \\ \alpha: K(x_2) \rightarrow K \\ \text{s.t. } R\text{-derivatives} \\ \in \mathcal{O}_{Z, x_0} \end{array} \right\}$

Suppose Set
 $T \rightarrow X$.

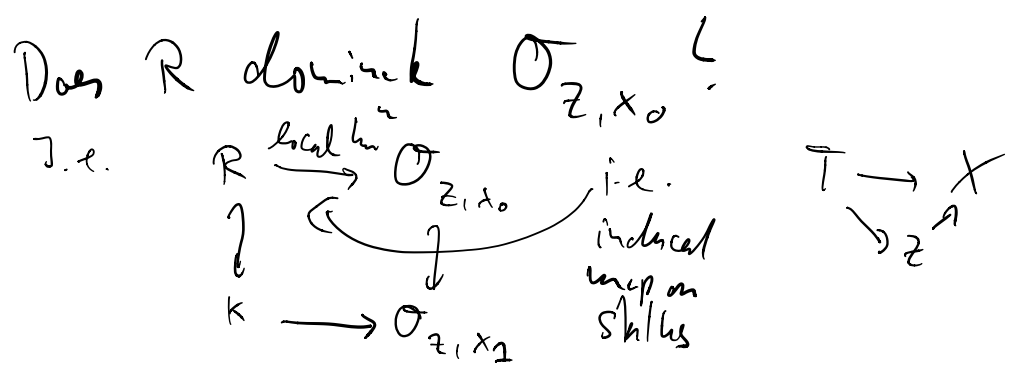
$x_0 \in \overline{\{x_2\}}$
 $\text{Map } \nu: \mathcal{O}_{Z, x_0}$

$$\begin{aligned}
 y_0 &\longmapsto x_0 \\
 y_2 &\longmapsto x_2 \\
 k(x_2) &\xrightarrow{\alpha} k(y_2) = K.
 \end{aligned}$$

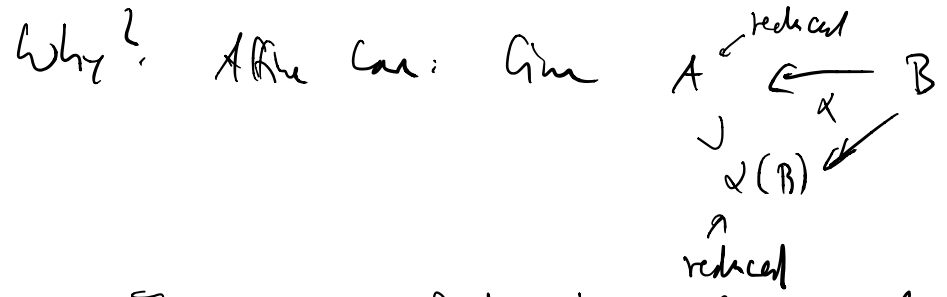
LA cb. map plus. Specializable.

$f: X \rightarrow Y$ cb. map.
 $x_2 \in X, x_0 \in \overline{\{x_2\}}$.
 $y_i := f(x_i)$.

Let $Z \subset Y$ closed, $x_2 \in Z$.
 Need: $x_2 \in Z$. $f^{-1}(Z) \subset X$ closed
 $x_2 \in f^{-1}(Z)$
 $\therefore x_0 \in f^{-1}(Z)$
 $\therefore y_0 \in Z \quad \square$



General fact: Give $X \xrightarrow{f} Y$ cov. of spec, X reduced
 let $X \rightarrow \overline{f(X)} \hookrightarrow Y$.



J.e. produce factorization through closed immersion.

Exercise: $\text{Spec } \alpha(B) = f(\text{Spec } A)$.

\therefore well-defined map

injectivity $f_1, f_2: T \longrightarrow X$

$F(f_1) = F(f_2)$

$f_1(y_i) = f_2(y_i) \quad f_1(x_0) = x_0$

The f_i factors through $\text{Spec } \mathcal{O}_{X, x_0}$ uniquely.

\therefore WMA $X = \text{Spec } A$ A local ring

$f_1(x_0) = \text{closed pt of } A$

$f_2(x_2) = P$ same prime ideal.

$$\begin{array}{ccc} R & \hookrightarrow & K \\ \uparrow f_1^\# & & \uparrow \alpha \\ A & \longrightarrow & k(P) \end{array} \quad \therefore f_1^\# = f_2^\#$$

$\therefore f_1 = f_2$.

Surjectivity

Suppose $p \in$

$x_2 \in \mathcal{A}$
 $x_0 \in \overline{\{x_2\}}$

$\alpha: k(x_0) \rightarrow k$

R dominates \mathcal{O}_{Z, x_0} .

Need to find $f: T \rightarrow X$ s.t. $F(f) = (k, \alpha)$.

Find. $T \xrightarrow{?} Z \hookrightarrow X$

$$\begin{array}{ccc} K & \xleftarrow{\alpha} & k(x_0) \\ \uparrow & & \uparrow \\ \mathbb{R} & \xleftarrow{\varphi} & \mathcal{O}_{Z, x_0} \end{array}$$

This does the job.



exists s.c. dominations
is a local homomorphism