X scheme

$\mathrm{QCoh}(X)$ = quasicoh. sheaves of $\mathcal{O}_X$-mod.

$\rotatebox{90}{$\in$}$

$\mathcal{O}_X$ commutative algebra object

$\mathcal{A}$ commutative $\mathcal{O}_X$-algebra object

$$\left(\begin{array}{l} \mathcal{O}_X \to \mathcal{A} \\ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \end{array} \right)$$

$\rightsquigarrow$ $\underline{\mathrm{Spec}}\, \mathcal{A}$ = a scheme with a map to X

1. Construct $\underline{\mathrm{Spec}}\, \mathcal{A}$.

Suppose $X = \mathrm{Spec}\, B$ qcoh. $\mathcal{O}_X$-mod $\simeq$ B-mod

qcoh. $\mathcal{O}_X$-alg $\simeq$ B-alg. $\ni A$

alg. str. $\to$ $B \to A$

$$\begin{tikzcd} \mathrm{Spec}\, A =: \underline{\mathrm{Spec}}\, \mathcal{A} \arrow[d, "\text{coming from algebra structure}"'] \\ X = \mathrm{Spec}\, B \end{tikzcd}$$

Basic idea of $\underline{\mathrm{Spec}}\, \mathcal{A}$

$$\begin{tikzcd} \underline{\mathrm{Spec}}\, \mathcal{A} \arrow[d] & ? \arrow[l] \arrow[d] \\ X & Y \arrow[l] \end{tikzcd}$$

$? \rightsquigarrow$ will be $\underline{\mathrm{Spec}}\, i^*\mathcal{A}$

in part. if $Y = \mathrm{Spec}\, B \underset{\text{open}}{\hookrightarrow} X$, then $\underline{\mathrm{Spec}}\, \mathcal{A} \times_X Y$

$= \operatorname{Spec} B$

$\rightsquigarrow$ $\underline{\operatorname{Spec}}\, \mathcal{A}$ is glued together from $\operatorname{Spec} B$ constructions.

How to do this systematically?

Functor of points formalism.

Basis: $\underline{\text{Yoneda lemma}}$ If $\mathcal{C}$ is any category,

the functor $\mathcal{C} \longrightarrow PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$

$$c \longmapsto R_c, \quad R_c(d) = Hom(d, c)$$

$$\Big[\, d \xrightarrow{\alpha} e \rightsquigarrow \underbrace{Hom(e, c) \xrightarrow{\circ\alpha} Hom(d, c)}\Big]$$

is fully faithful.

$$Sch_X \hookrightarrow PSh(Sch_X)$$

$$\ni F$$

Write down F. Does it come from a scheme?

"Is it a representable functor?"

$\underline{\text{Lemma}}$ F is representable $\iff$ (1) $\exists$ open cover $\{U_i\}$ of X

s.t. $F \times_{R_X} R_{U_i}$ is rep. $\forall i$

$\in PSh(Sch_X)$

(2) F is a sheaf. [let's ignore this]

Def'n $\quad F = \underline{\mathrm{Spec}}\,\mathcal{A}$

$$F(T) = \text{"}\mathrm{Hom}_X(T, \underline{\mathrm{Spec}}\,\mathcal{A})\text{"} \qquad \pi : T \longrightarrow X$$

$$= \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, \pi_*\mathcal{O}_T)$$

$X = \mathrm{Spec}\, B \qquad \mathcal{A} \longleftrightarrow A$

$\underline{\mathrm{Spec}}\,\mathcal{A} = \mathrm{Spec}\, A$

$$\mathrm{Hom}_B(T, \mathrm{Spec}\, A) = \mathrm{Hom}_{B\text{-alg}}(A, \mathcal{O}_T(T))$$

$$\text{e.g. } \mathrm{Hom}_k(X, \mathbb{A}^1_k) = \mathcal{O}_X(X)$$

To check (1): $X \hookleftarrow \mathrm{Spec}\, B = Y$

$$\begin{tikzcd}
 & \mathrm{Spec}\, A \overset{?}{=} P \arrow[r] \arrow[d] & F \ \underline{\mathrm{Spec}}\,\mathcal{A} \arrow[d] \\
R_T \arrow[ur, dashed, "?"] \arrow[r, bend right, "p"'] & \mathrm{Spec}\, B = R_Y \arrow[r, "i"] & R_X \arrow[d, equal] \\
 & & X
\end{tikzcd}$$

what is P?

really: is it a scheme?

$\left[\mathrm{Hom}_{PSh}(R_T, F) \simeq F(T)\right]$

$$\mathrm{Hom}_{PSh(Sch_X)/R_Y}(R_T, P) = \mathrm{Hom}_{PSh(Sch_X)}(R_T, F)$$

$$= \mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(\mathcal{A}, i_*p_*\mathcal{O}_T)$$

$$= \mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(i^*\mathcal{A}, p_*\mathcal{O}_T)$$

$$= \mathrm{Hom}_{B\text{-alg}}(A, \mathcal{O}_T(T))$$

$$= \mathrm{Hom}_Y(T, \mathrm{Spec}\, A)$$

$\therefore \mathrm{Hom}_{PSh(Sch_X)}(h_T, P) = \mathrm{Hom}_X(T, \mathrm{Spec}\, A)$

<u>Summary</u>: $\underline{\mathrm{Spec}} : \left(\mathcal{O}_X^{q.coh.}\text{-alg}\right)^{op} \longrightarrow PSh(Sch_X)$

$$\mathcal{A} \longmapsto \left(T \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{O}_T)\right)$$

$$\begin{array}{c} PSh(Sch_X) \\ \uparrow \\ \big\downarrow \text{Yoneda} \\ Sch_X \end{array}$$

takes values in the ess. image of $\longrightarrow Sch_X$.

(2) Show: $\underline{\mathrm{Spec}}$ is an equivalence

$$(q.coh.\ \mathcal{O}_X\text{-alg.})^{op} \simeq (\text{affine } X\text{-schemes})$$

$$\begin{tikzcd} (q.coh.\ \mathcal{O}_X\text{-alg.})^{op} \arrow[dr] & (\text{affine } X\text{-schemes}) \arrow[d, hook] \\ & (X\text{-schemes}) \end{tikzcd}$$

- check that $\begin{array}{c}\underline{\mathrm{Spec}\,\mathcal{A}} \\ \downarrow \\ X\end{array}$ is affine

- exhibit "inverse" functor

$$G : (\text{affine } X\text{-schemes}) \longrightarrow (\mathcal{O}_X^{q.coh.}\text{-alg.})$$

s.t. $\quad G \circ \underline{\mathrm{Spec}} \simeq id_{q.coh.\ \mathcal{O}_X\text{-alg}}$

$$\underline{\mathrm{Spec}} \circ G \simeq id_{(\text{affine } X\text{-schemes})}$$

$$\begin{array}{ccc} & \mathrm{Spec}\,\mathcal{A} \mapsfrom \mathcal{A} & \\ Y & & \\ \text{affine } X\text{-sch}: \ \pi \downarrow & \xmapsto{G} & \pi_* \mathcal{O}_Y \\ X & & \end{array}$$

... proof omitted ...

(3) deduce that (closed subsch of X) $\simeq$ $\left(\begin{array}{c}\text{q coh. ideal} \\ \text{sheaves} \\ \text{of } \mathcal{O}_X \end{array} \right)$

$\|$

affine X-sch $Y \xrightarrow{i} X$

s.t. $\mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Y$

$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{A} \to 0$

i.e. (closed subsch.) = (quotients of $\mathcal{O}_X$ which are q coh.)

= (q coh. ideal sheaves)