normalization $\tilde{X}$
$\downarrow$
X ← integral scheme

- birational morphism
- $\tilde{X}$ normal
- universal w.r.t. the above

integral ⇐ normal ⇐ regular

⇒ curves

for curves: normalization = desingularization

What about higher dimensions?

normal $\not\Rightarrow$ regular

Blowing up: $X \hookleftarrow Z$ $\quad Bl_Z X$

$$\begin{array}{c} Bl_Z X \\ \downarrow \\ X \end{array}$$

← birational morphism (iso outside of Z)

ideal question: Can you make X regular by blowing up many times?

curves: yes, blow up singular points repeatedly

answer is yes if: over a perfect field, dim ≤ 3 (Zariski)
char. = 0 (Hironaka)

Category $\mathcal{C}$, class of morphisms W

Look at pairs $(\mathcal{D}, \alpha: \mathcal{C} \to \mathcal{D})$ s.t. α sends W to isomorphisms.
"α inverts W"

Want there to be a "universal example".

I.e. $\alpha: \mathcal{C} \to \mathcal{C}[W^{-1}]$

s.t. $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\alpha^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$

is fully faithful onto those functors inverting W

Difficulties: 1) make this precise
2) does $\mathcal{C}[W^{-1}]$ exist (will be well-defined up to canonical equiv.)

Gabriel–Zisman
Adams

Try to answer 2) as follows:

Say that objects of $\mathcal{C}[W^{-1}]$ = objects of $\mathcal{C}$

mor. in $\mathcal{C}[W^{-1}]$ from X to Y are strings

$$\left\{ X \to X_1 \xleftarrow{W} X_2 \to X_3 \xleftarrow{W} X_4 \to \cdots \to Y \right\} \big/ \sim$$

Problem: - not clear this will be a set
- completely impossible to work with

$\mathcal{A}$ abelian category

$D(\mathcal{A}) = C(\mathcal{A})[(\text{quasi-isos})^{-1}]$

$K(\mathcal{A})$ = chain cxes with morphisms up to chain htpy

theorem: $\exists$ subcat. $\mathcal{C}$ s.t. $\mathcal{C}$ is equivalent to $D(\mathcal{A})$

e.g. $\mathcal{C}$ contains bounded complexes of injectives or projectives (can infinite in correct dir^n)

$A \in \mathcal{C}$ $\quad A \to I_1 \to I_2 \to I_3 \to \cdots$

injective res^n

$\cdots \to P_2 \to P_1 \to A$ projective res^n

"homotopical algebra"

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1. rat^l curve$/k$ = $\quad$ curve$/k$ = f.t., sep., integral

rat^l = birational to $\mathbb{A}^1$

rat^l non-singular curve = $\mathbb{P}^1, \mathbb{A}^1, U \subset \mathbb{A}^1$ open

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$X \hookrightarrow X'$ proper + non-singular $\dashrightarrow \mathbb{P}^1$ $\quad \exists!$ by problem sheet 6

U open

$U \xhookrightarrow{\text{open imm.}} \mathbb{A}^1$ open imm.?

← by pbs 2 sheet 6

$\#$

$\emptyset$

2. char $k \neq 2$

$Y = \operatorname{Spec} k[Y] \qquad k[Y] = k[x,y]/y^2-(x^3-x)$

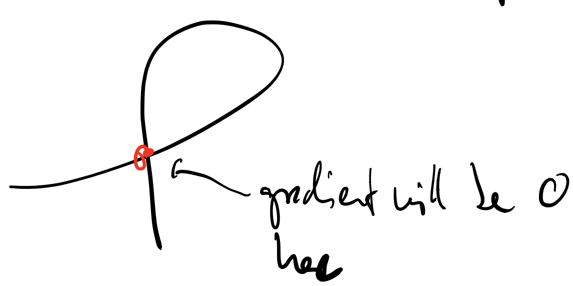
$\hookrightarrow \mathbb{A}^2$

(1) Y is non-singular.

the tangent space to Y at some point $p \in Y$

$f = c$ grad f is a vector direction of steepest ascent

tangent line is orthogonal to grad f.



Upshot: Sing locus of Y
is given by grad $f = 0 \cap Y$

$$\operatorname{grad} f = \begin{pmatrix} -3x^2+1 \\ 2y \end{pmatrix} = 0 \qquad y^2 = x^3 - x \Bigg\} \quad \begin{array}{l} y = 0 \quad (\text{char.} \neq 2) \\ x = x^3 \quad x(1-x^2) = 0 \end{array}$$

$x=0$ ✗ $\quad -3x^2+1 = 1 \neq 0$

$x^2 = 1$ ✗ $\quad -3x^2+1 = -2 \neq 0$

$\therefore Y$ is non-singular

(5) $k[Y]$ is a domain: (= integral domain)

$$k[Y] = \frac{k[x,y]}{(y^2 - x^3 + x)}$$

i.e. if $P, Q \in k[x,y]$, $\bar{P} \cdot \bar{Q} = 0 \in k[Y]$ then $\bar{P} = 0$ or $\bar{Q} = 0$

i.e. if $PQ \in (y^2 - x^3 + x)$ then P or $Q \in$ — this ideal is prime

i.e., since $k[x,y]$ a UFD, need $y^2 - x^3 + x$ to be irreducible

think of this as a polynomial in y, with coeff in $k[x]$

$$y^2 + (x - x^3)$$

- monic of degree 2. Can only factor as product of 2 monic linear factors

$$(y + p(x))(y + q(x)) = y^2 - x^3 + x$$

$$= y^2 + y(p(x) + q(x)) + p(x) \cdot q(x) = y^2 - q^2$$

$\therefore p = -q$

iff $x^3 + x = q^2$

impss. by degrees //

(3) $R \subset k[Y]$ subring gen. over k by x.

Show: $R \cong k[x]$. I.e. $k[x] \longrightarrow k[Y]$, $x \longmapsto x$

should be injective.

$$k[x] \hookrightarrow k[x,y] \longrightarrow k[Y] = \frac{k[x,y]}{(y^2-x^3+x)}$$

kernel = ideal (y^2-x^3+x)

$P(x) \longmapsto 0 \iff P(x) \in (y^2-x^3+x)$

impossible unless $P=0$ because

$\deg_y P(x) = 0 \quad \deg_y(y^2-x^3+x) = 2$

(4) Construct $\sigma : k[Y] \xrightarrow{\sim} k[Y]$

s.t. $\sigma(x) = x$

$\sigma(y) = -y$

$$y^2-x^3+x \longmapsto (-y)^2 - x^3+x = y^2-x^3+x$$

$$\begin{tikzcd} k[x,y] \arrow[d] \arrow[dr, "x \mapsto x,\ y \mapsto -y"] & \\ k[Y] = \frac{k[x,y]}{(y^2-x^3+x)} \arrow[r, dashed, "\exists ?"] & k[Y] \end{tikzcd}$$

$= 0$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \\ A/I \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

iff $\varphi(I) = 0$

$\vdots$

$y^2 = x^3 - x = x(x^2-1)$

$\therefore x \mid y^2$ one may show: x, y are irred. & $x \nmid y$

$\therefore k[Y]$ is not a UFD

BUT any affine rat'l curve is $\mathbb{A}^1 \setminus \{$finitely many pts$\}$

$\therefore$ coord. ring = localization of $k[x]$

i.e. a UFD