Algebraic Geometry 2

Exercises Tutorium 4

Dr. Tom Bachmann Winter Semester 2020–21

Exercise 1. Let A be an integrally closed, noetherian integral domain with fraction field K, and L/K a finite separable extension. Let B be the integral closure of A in L. Show that B is a finite type A-module, as follows:

(1) Show that the trace form $\varphi : L \to K$ maps B into A.
(2) Show that there exist $b_1, \ldots, b_n \in B$ forming a basis of L as a K-vector space.
(3) Conclude by proving that $B \subset \{x \in L \mid \varphi(xb_i) \in A \text{ for all } i\} \simeq A^n$.

Exercise 2. Let $\mathcal{P}$ be a collection of morphisms of schemes containing the closed immersions and being stable under composition and base change. Show that (a) products of morphisms in $\mathcal{P}$ are in $\mathcal{P}$, (b) if $g \circ f \in \mathcal{P}$ and g is separated, then $f \in \mathcal{P}$, and (c) if $f \in \mathcal{P}$ then $f_{red} \in \mathcal{P}$.

Deduce that projective morphisms satisfy (a), (b), (c).

Exercise 3. Show that an integral domain R is a valuation ring if and only if given ideals I, J we have either $I \subset J$ or $J \subset I$. Deduce that if R is a valuation ring and P is a prime ideal, then R/P and R_P are valuation rings.