1. $A$ noeth. integral domain
integrally closed in $K = \mathrm{Frac}(A)$
$L/K$ finite sep.
$B$ = integral closure of $A$ in $L$
Goal: prove that $B$ is f.g. as $A$-mod.
(1) Show that $\varphi(B) \subset A$ the trace
$\varphi : L \to K$ form.
$\varphi = tr_{L/K}$

$L/K$ finite ext$^n$
$x \in L \rightsquigarrow m_x : L \to L$ is $K$-linear
$y \mapsto y \cdot x$
$tr_{L/K}(x) = tr\, m_x$

FACTS: • If $P$ is the min. poly of $x$
then $tr(x) = \pm$ the 2$^{nd}$ highest coeff. of $P$
$P(T) = T^n \pm tr(x) T^{n-1} + \dots$
• $L/K$ is sep $\Leftrightarrow$ $tr$ is non-degenerate
i.e. $L \otimes_K L \to K$
$x \otimes y \mapsto tr(xy)$
is a perfect pairing
i.e. $L \xrightarrow{\simeq} \mathrm{Hom}_K(L, K)$
$x \longmapsto (y \mapsto tr(xy))$

STP: if $x \in B$, $P$ the min. poly, then all coeff. $P$

$\sim$ $\sigma$

$B$ lie in $A$.

With $P(T) = (T - x_1)(T - x_2) \dots (T - x_n)$

$x_1 = x \qquad x_i \in \overline{K}$

Claim: each $x_i$ is integral over $A$

$\longrightarrow$ sv. s.t. $P(x_i) = 0$

Then $\mathrm{tr}(x) = x_1 + x_2 + \dots + x_n$ integral over $A$

& $\in K$

$\vdots$

$\therefore \in A$

$\det(x) = \prod x_i$

(2) Show that $\exists\, b_1, \dots, b_n \in B$ forming a $K$-basis of $L$.

$L/K$ finite ext$^n$, basis $x_1, \dots, x_n \in L$.

Issue: $\overset{x}{\underset{\shortparallel}{}}x_i$ may not be integral $/A$.

Suppose $x^m + y_1 x^{m-1} + \dots + y_m = 0 \qquad y_i \in K$

Let $c \in K$.

Mult. by $c^m$: $(cx)^m + y_1 c\,(cx)^{m-1} + \dots + c^n y_m = 0$

Want: $cx$ integral $/A$ i.e. $y_i \cdot c^i \in A$

Know $K = \mathrm{Frac}(A)$, i.e. $y_i = \frac{a_i}{a_i'} \rightsquigarrow c = \prod a_i'$.

(3) Conclude by proving that

(a) $B \subset M = \{x \in L \mid \varphi(x b_i) \in A\ \forall i\} \cong A^n$.

(a), assuming (b) & (c): $\overset{(c)}{A^n}$ is f.t. $A$-module

$\overset{(b)}{\phantom{x}}$

$\cup$

$B$ $\therefore B$ is f.t. b/c $A$ is noeth.

(c): $\varphi$ is non-deg. $\rightsquigarrow$ $\exists$ dual basis $v_1, \dots, v_n \in L$

i.e. $\varphi(v_i \cdot s_j) = \begin{cases} 1 & : i=j \\ 0 & : \text{else} \end{cases}$.

$x \in L \rightsquigarrow x = \sum_i c_i \cdot v_i \quad c_i \in K$.

$\varphi(x \cdot s_j) = \sum_i \varphi(c_i \cdot v_i \cdot s_j) = c_j$

$\therefore x \in M \iff c_i \in A \;\forall i$

$$\begin{array}{ccccc} \therefore & M & \cong & A^n & = \oplus A \cdot v_i \\ & \downarrow & & \downarrow & \\ & L & \cong & K^n & = \oplus K \cdot v_i \end{array}$$

(b) $x \in B \Rightarrow \varphi(x \cdot s_i) \in \varphi(B) \overset{(1)}{\subset} A$ $\square$

2. $P$ coll. of mor. of schemes

- {closed im.} $\subset P$
- $P$ stable under comp.
- $P$ stable under base change

(5) $g \circ f \in P$, $g$ sep $\Rightarrow$ $f \in P$

$$X \xrightarrow{f} Y \underset{\text{sep.}}{\xrightarrow{g}} Z$$

$$\begin{tikzcd}
X \arrow[r, "\therefore P"] \arrow[d] & X \times_Z Y \arrow[d] \arrow[r, hook] & X \arrow[dr, "P"] & \\
Y \arrow[r, hook, "P"'] & Y \times_Z Y & & Z \\
& & Y \arrow[ur] &
\end{tikzcd}$$

$$X \times_Z Y \xrightarrow{\therefore P} Y$$

(6) $f \in P \Rightarrow f_{red} \in P$

$$\begin{tikzcd}
X \arrow[r, "f", "\in P"'] & Y \\
X_{red} \arrow[u, hook, "P"] \arrow[r, dashed, "\exists! f_{red}"] & Y_{red} \arrow[u, hook, "\text{c.i. } \therefore \text{sep.}"']
\end{tikzcd}$$

$P$ ; $\therefore \in P$

$\therefore f_{red} \in P$

$$\begin{tikzcd}
B \arrow[d] & A \arrow[l] \arrow[d] \\
B/N(B) & A/N(A) \arrow[l, dashed]
\end{tikzcd}$$

red

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr, dashed, "\exists!"'] & X \\
& X_{red} \arrow[u, hook]
\end{tikzcd}$$

red.

$$\begin{tikzcd}
C & A \arrow[l] \arrow[d] \\
& A/N(A) \arrow[ul, dashed, "\exists!"]
\end{tikzcd}$$

3. $R$ integral domain.

Show: $R$ a val$^n$ ring $\Longleftrightarrow$ given ideals $I, J$,

(i.e. $x \in \operatorname{Frac}(R)$
$\Rightarrow$ either $x \in R$
or $x^{-1} \in R$)

either $I \subset J$ or $J \subset I$.

$\Leftarrow$: $x = \frac{a}{b} \in \operatorname{Frac}(R)$ want: $a \mid b$ or $b \mid a$

i.e. $b \in (a)$ or $a \in (b)$

i.e. $(b) \subset (a)$ or $(a) \subset (b)$

$\Rightarrow$: Suppose $a \in I \setminus J$. Want: $J \subset I$.

Let $b \in J$, put $x = \frac{a}{b}$.

If $x \in R$ then $a = x \cdot b \in J$ $\times$.

$\therefore$ $x^{-1} \in R$ & $b = x^{-1} \cdot a \in I$ $\square$