

1.  $A$  noeth. integral domain  
 integrally closed in  $K \simeq \text{Frac}(A)$

$L/K$  finite sep.

$B =$  integral closure of  $A$  in  $L$

Goal: prove that  $B$  is f.g. as  $A$ -mod.

(1) Show that  $\varphi(B) \subset A$  the trace form.

$$\varphi: L \rightarrow K \text{ form.}$$

$L/K$  finite ext<sup>n</sup>  $\text{tr}_{L/K}$

$x \in L \rightsquigarrow m_x: L \rightarrow L$  is  $K$ -linear  
 $y \mapsto y \cdot x$

$$\text{tr}_{L/K}(x) = \text{tr } m_x$$

FACTS: If  $P$  is the min. poly of  $x$   
 then  $\text{tr}(x) = \pm$  the  $2^{\text{nd}}$  highest coeff. of  $P$

$$P(T) = T^n \pm \text{tr}(x) T^{n-1} + \dots$$

$L/K$  is sep  $\Leftrightarrow$   $\text{tr}$  is non-degenerate

$$\text{i.e. } L \otimes_K L \rightarrow K$$

$$x \otimes y \mapsto \text{tr}(xy)$$

is a perfect pairing

$$\text{i.e. } L \xrightarrow{\cong} \text{Hom}_K(L, K)$$

$$x \mapsto (y \mapsto \text{tr}(xy))$$

STP: if  $x \in B$ ,  $P$  the min. poly, then all coeff.  $P$

$B$  lie in  $A$ .

$$\text{With } P(T) = (T - x_1)(T - x_2) \dots (T - x_n)$$

$$x_1 = x \quad x_i \in \overline{K}$$

Claim: each  $x_i$  is integral over  $A$

$$\rightarrow \text{ev. s.t. } P(x_i) = 0$$

Then  $\text{tr}(x) = x_1 + x_2 + \dots + x_n$  integral over  $A$

$$\vdots \quad \& \in K$$

$$\det(x) = \prod x_i \quad \therefore \in A$$

(2) Show that  $\exists b_1, \dots, b_n \in B$   
forming a  $K$ -basis of  $L$ .

$L/K$  finite ext<sup>n</sup>, basis  $x_1, \dots, x_n \in L$ .

(since:  $x_i$  may not be integral /  $A$ .)

$$\text{Suppose } x^m + \gamma_1 x^{m-1} + \dots + \gamma_m = 0 \quad \gamma_i \in K$$

Let  $c \in K$ .

$$\text{Mult. by } c^m: (cx)^m + \gamma_1 c (cx)^{m-1} + \dots + c^m \gamma_m = 0$$

Want:  $c \in A$  integral /  $A$  i.e.  $\gamma_i = c^i \in A$

$$\text{Know } K = \text{Frac}(A), \text{ i.e. } \gamma_i = \frac{a_i}{d_i} \rightarrow c = \prod a_i$$

(3) Conclude by proving that

$$B \subset M = \{x \in L \mid \varphi(x, s_i) \in A \forall i\} \cong A^n$$

(a), assuming (b) & (c):  $A^n$  is f.t.  $A$ -module  
 $\mathcal{B}$  is f.t.  $\mathcal{L}$  is noth.

(c):  $\varphi$  is non-deg.  $\leadsto \exists$  dual basis  $v_1, \dots, v_n \in \mathcal{L}$   
 i.e.  $\varphi(v_i \cdot s_j) = \begin{cases} 1 & : i=j \\ 0 & : \text{else} \end{cases}$

$x \in \mathcal{L} \leadsto x = \sum_i c_i \cdot v_i \quad c_i \in K.$

$$\varphi(x \cdot s_j) = \sum_i c_i \varphi(v_i \cdot s_j) = c_j$$

$\therefore x \in \mathcal{M} \Leftrightarrow c_i \in A \quad \forall i$

$\therefore \mathcal{M} \cong A^n = \bigoplus A \cdot v_i$

$\downarrow \quad \downarrow$   
 $\mathcal{L} \cong K^n = \bigoplus K \cdot v_i$

(b)  $x \in \mathcal{B} \Rightarrow \varphi(x \cdot s_i) \in \varphi(\mathcal{B}) \stackrel{(1)}{\subset} A \quad \square$

2.  $\mathcal{P}$  coll. of var. of schemes

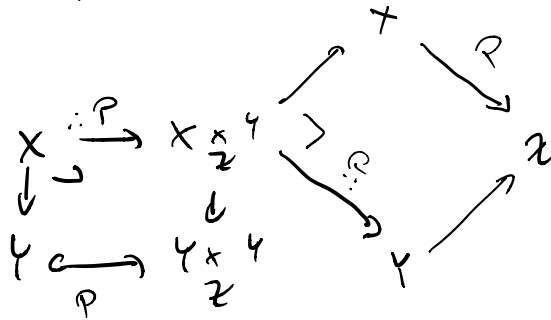
-  $\{\text{closed in.}\} \subset \mathbb{C}P$

-  $\mathcal{P}$  stable under  $\text{grp}^n$

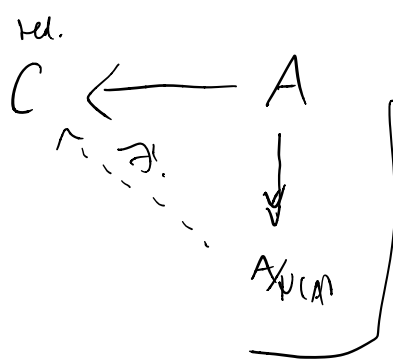
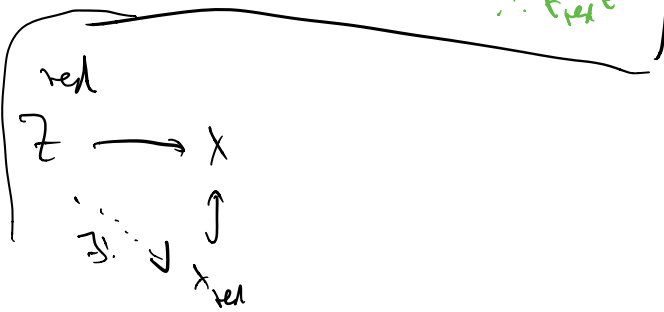
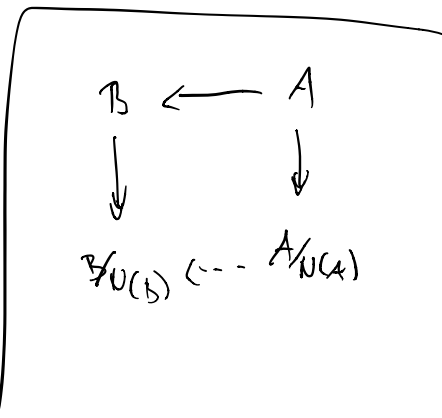
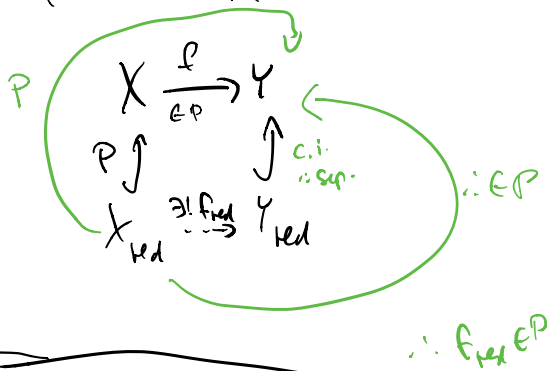
-  $\mathcal{P}$  stable under base change

(5)  $g \circ f \in \mathcal{P}$  &  $\text{sep} \Rightarrow f \in \mathcal{P}$

$$X \xrightarrow{f} Y \xrightarrow[\text{sep.}]{g} Z$$



(6)  $f \in \mathcal{P} \Rightarrow f_{\text{rel}} \in \mathcal{P}$



3.  $R$  integral domain.

Show:  $R$  a val<sup>s</sup> ring  $\Leftrightarrow$  given ideals  $I, J,$

(i.e.  $x \in \text{Frac}(\mathbb{R})$ )  
 $\Rightarrow$  either  $x \in \mathbb{R}$   
or  $x' \in \mathbb{R}$ )

either  $\mathbb{I} \subset \mathbb{J}$  or  $\mathbb{J} \subset \mathbb{I}$ .

$\Leftarrow$ :  $x = \frac{a}{s} \in \text{Frac}(\mathbb{R})$

want:  $a|s$  or  $s|a$

i.e.  $s \in (a)$  or  $a \in (s)$

i.e.  $(s) \subset (a)$  or  $(a) \subset (s)$

$\Rightarrow$ : Suppose  $a \in \mathbb{I} \setminus \mathbb{J}$ . Want:  $\mathbb{J} \subset \mathbb{I}$ .

Let  $s \in \mathbb{J}$ , put  $x = \frac{a}{s}$ .

$\exists x \in \mathbb{R}$  then  $a = x \cdot s \in \mathbb{J}$  ~~X~~.

$\therefore x' \in \mathbb{R}$  &  $s = x' \cdot a \in \mathbb{I}$   $\square$