Algebraic Geometry 2

Exercises Tutorium 2

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Exercise 1. Let $S_\bullet, T_\bullet$ be commutative $\mathbb{N}$-graded rings with $S_0 = T_0 = A$. Put

$$R_\bullet = \bigoplus_{d \geq 0} S_d \otimes_A T_d.$$

Show that $R_\bullet$ is a commutative $\mathbb{N}$-graded ring, and construct an isomorphism

$$\mathbf{Proj}(R_\bullet) \simeq \mathbf{Proj}(S_\bullet) \times_{\mathrm{Spec}\,(A)} \mathbf{Proj}(T_\bullet).$$

Deduce that there is a closed immersion

$$\mathbb{P}^n_A \times_A \mathbb{P}^m_A \hookrightarrow \mathbb{P}^N_A.$$

Exercise 2. A morphism $X \to Y$ is called *projective* if there exists a closed immersion of Y-schemes $X \hookrightarrow \mathbb{P}^n_Y$ for some n. Show that projective morphisms are stable under composition.

Exercise 3. Let $\mathcal{P}$ be a collection of morphisms of schemes containing the closed immersions and being stable under composition and base change. Show that (a) products of morphisms in $\mathcal{P}$ are in $\mathcal{P}$, (b) if $g \circ f \in \mathcal{P}$ and g is separated, then $f \in \mathcal{P}$, and (c) if $f \in \mathcal{P}$ then $f_{red} \in \mathcal{P}$.

Deduce that projective morphisms satisfy (a), (b), (c).

Exercise 4. Show that an integral domain R is a valuation ring if and only if given ideals I, J we have either $I \subset J$ or $J \subset I$. Deduce that if R is a valuation ring and P is a prime ideal, then R/P and R_P are valuation rings.