1. $S_\cdot$, $T_\cdot$ graded rings, $S_0 = A = T_0$.

$$R_\cdot = \bigoplus_{d \geq 0} S_d \otimes_A T_d.$$

Show: $\operatorname{Proj}(R_\cdot) \simeq \operatorname{Proj}(S_\cdot) \times_{\operatorname{Spec}(A)} \operatorname{Proj}(T_\cdot)$

$\operatorname{Proj}(S \otimes_A T) \overset{?}{=}$

No!

"Reason":

$\operatorname{Proj}(S) = \operatorname{Spec} S^{\setminus 0} / \mathbb{G}_m$

$\operatorname{Proj}(S \otimes_A T) = \operatorname{Spec}(S \otimes_A T)^{\setminus 0} / \mathbb{G}_m$

$= \dfrac{\operatorname{Spec}(S) \times_A \operatorname{Spec}(T)}{\mathbb{G}_m \times \mathbb{G}_m}$

$\neq \dfrac{\operatorname{Spec} S}{\mathbb{G}_m} \times \dfrac{\operatorname{Spec} T}{\mathbb{G}_m}$

Try again: $\operatorname{Proj}(R_\cdot) = \bigcup_{\substack{f \in R_\cdot \\ \text{homog.}}} \underbrace{D_+(f)}_{= \operatorname{Spec}(R_{(f)})}$

$R_{(f)}$: degree 0 part of R_f

$\operatorname{Proj}(S) \times_A \operatorname{Proj}(T) = \bigcup_{\substack{f \in S \\ g \in T \\ \text{hmg.}}} D_+(f) \times_A D_+(g)$

Suppose f, g hmg. $f \otimes g \in R$ $\quad D_+(f \otimes g)$

Try to prove: if $f \in S_d$, $g \in T_d$ then $D_+(f) \times_A D_+(g) \cong D_+(f \otimes g)$.

Equivalently $(S_f)_0 \otimes_A (T_g)_0 \xrightarrow{\text{build map}} \cong (R_{f \otimes g})_0$

(intuition: f, g deg. 1 ... elt $\frac{s}{f^n}$, s of deg. n, since $\frac{t}{g^n}$ ... $\frac{s \otimes t}{(f \otimes g)^n}$)

Trick: $(S_f)_0 = \operatorname{colim}(S_0 \xrightarrow{f} S_1 \xrightarrow{f} S_2 \to \cdots)$

$s \in S_n \mapsto \frac{s}{f^n}$, into $S_{(f)}$

$\therefore \quad S_{(f)} \otimes_A T_{(g)} = \operatorname{colim}(S_{(f)} \otimes T_0 \to S_{(f)} \otimes T_1 \to \cdots)$

$$= \operatorname{colim} \begin{pmatrix} S_0 \otimes T_0 & \xrightarrow{f} & S_0 \otimes T_1 & \to \cdots \\ \downarrow g & & \downarrow & \\ S_1 \otimes T_0 & \to & S_1 \otimes T_1 & \to \cdots \\ \downarrow & & \downarrow & \\ \vdots & & \vdots & \end{pmatrix}$$

$= \mathrm{colim}\,(S_0 \otimes T_0 \xrightarrow{f\otimes g} S_1 \otimes T_1 \xrightarrow{f\otimes g} \cdots)$

$= R_{(f\otimes g)}$

Back to $\mathrm{Proj}(S_\cdot) \times_A \mathrm{Proj}(T) \overset{?}{\cong} \mathrm{Proj}(R)$

Know: $f, g \in S, T$ same degree

$$\begin{array}{ccc} \cup & & \cup \\ D_+(f) \times_A D_+(g) & \cong & D_+(f \otimes g) \end{array}$$

compatible

$\therefore$ If $U \subset \mathrm{Proj}(S) \times_A \mathrm{Proj}(T)$ is $\bigcup D_+(f) \times_A D_+(g)$ f, g same deg.

$V \subset \mathrm{Proj}(R)$ $\bigcup D_+(f \otimes g)$ same deg.

then $U \cong V$.

either equality?

Complement of V $\leftrightarrow$ ideal $\langle f\otimes g \mid f \in S_d, g \in T_d \rangle$

$= R_+ = \bigoplus_{d>0} R_d$

"irrelevant ideal"

$\therefore V = \mathrm{Proj}(R)$

Proj (S) is covered by $D_+(f)$ f homog. of the degree

Proj (T) — " $D_+(g)$, g

Recall that $D_+(f) = D_+(f^n)$ $\forall n$

$\therefore$ replace f, g (pos. diff. deg.)

by $f^{\deg g}$, $g^{\deg f}$ $\therefore$ $U = \mathrm{Proj}\, S \times_A \mathrm{Proj}\, T$.

Deduce that $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$.

$S = A[X_0, \ldots, X_n]$

$T = A[Y_0, \ldots, Y_m]$

$$\mathbb{P}^m_A \times_A \mathbb{P}^n_A = \mathrm{Proj}\, S \times_A \mathrm{Proj}\, T = \mathrm{Proj}\,(R) \overset{?}{\hookrightarrow} \mathbb{P}^N = \mathrm{Proj}\,(A[Z_0, \ldots, Z_N])$$

i.e. $A[Z_0, \ldots, Z_N] \longrightarrow R$.

i.e. choose $Z_i \longmapsto ? \in R_1 = S_1 \otimes T_1$

$$= A\{X_0, \ldots, X_n\} \otimes A\{Y_0, \ldots, Y_m\}$$

Name variables $Z_{ij} \longmapsto X_i \otimes Y_j$

$i = 0, \ldots, n$

$j = 0, \ldots, m$

$$N = (n+1)(m+1) - 1 = nm + n + m$$

map is surjective? $\iff$ $x_i \otimes y_j$ gen. R

$$R_d = S_d \otimes T_d$$

S_d elements are sums of products of $\underline{d}$ x_i-s

T_d — — — — y_i-s

with in product of d $x \otimes y$-s.

$$x_0^d \otimes y_0 y_1^{d-1} = (x_0 \otimes y_0) \cdot (x_0 \otimes y_1)^{d-1}$$

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2. Show that projective mor. are closed under comp^n.

$X \to Y$ proj.

$\iff$ $X \hookrightarrow \mathbb{P}^n_Y$ for some n.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[rrr, bend left, "?"] & \mathbb{P}^m_Y = \mathbb{P}^m \times Y \arrow[r, hook] \arrow[dl] & \mathbb{P}^m \times \mathbb{P}^n \times Z \arrow[r, hook] \arrow[dl] & \mathbb{P}^N_Z \arrow[ddlll] \\ Y \arrow[r, hook] \arrow[d] & \mathbb{P}^n_Z = \mathbb{P}^n \times Z \arrow[dl] & & \\ Z & & & \end{tikzcd}$$

ex. 1.

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3. $\mathcal{P}$ some class of mor. of Schemes s.t.

(1) $\{\text{closed imm.}\} \subset \mathcal{P}$

(2) $f, g \in \mathcal{P} \Rightarrow g \circ f \in \mathcal{P}$ (if f, g composable)

(3)
$$\begin{tikzcd} S \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd} \qquad f \in \mathcal{P} \Rightarrow f' \in \mathcal{P}.$$

Then: (4)
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr] & & Y' \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{array}{l} f, f' \in \mathcal{P} \\ \Rightarrow f \times_S f' \in \mathcal{P} \end{array}$$

(5) $g \circ f \in \mathcal{P}$ & g sep. $\Rightarrow f \in \mathcal{P}$

(6) $f : X \to Y \in \mathcal{P} \Rightarrow f_{red} : X_{red} \to Y_{red} \in \mathcal{P}$.

(4)
$$\begin{tikzcd} X \times_S X' \arrow[rr, "f \times f'", "\therefore \in \mathcal{P}"'] \arrow[dr, "id \times f'"'] & & Y \times_S Y' \\ & X \times Y' \arrow[ur, "f \times id"'] & \end{tikzcd}$$

$\in \mathcal{P} \Leftrightarrow$
$$\begin{tikzcd} X \times Y' \arrow[r] \arrow[d, "f \times id"'] & X \arrow[d, "f"] \\ Y \times Y' \arrow[r] & Y \end{tikzcd}$$

$\mathcal{P}$

(symm. arg.)