Algebraic Geometry 2

Exercises Tutorium 2

Dr. Tom Bachmann Winter Semester 2020–21

Exercise 1 (Recollections on reduced schemes)**.**

(1) Define reduced schemes.
(2) Given a scheme X and a closed subset $Y \subset X$, construct the *reduced closed subscheme* structure Y_{red} on Y.
(3) State and prove a universal property of Y_{red}.

Exercise 2. Establish the following properties of morphisms of schemes.

(1) Closed immersions are proper.
(2) Compositions of proper morphisms are proper.
(3) Proper morphisms are stable under base change.
(4) Products of proper morphisms are proper.
(5) If $g \circ f$ is proper and g is separated, then f is proper.
(6) Properness is local on the target.

Exercise 3. Let k be an algebraically closed field and $X \subset \mathbb{A}^n_k$ an affine variety which is *proper*. Show that X consists of finitely many points [*Hint:* first treat the case $n = 1$]. Deduce that a proper affine morphism of schemes is quasi-finite.

Remark: Proper affine morphisms are in fact *finite*. (But this requires more work.)

Exercise 4. Show that an integral domain R is a valuation ring if and only if given ideals I, J we have either $I \subset J$ or $J \subset I$. Deduce that if R is a valuation ring and P is a prime ideal, then R/P and R_P are valuation rings.