1. (1) $X \in \mathrm{Sch}$. is reduced

$\iff \mathcal{O}_X$ has no nilpotents

$\iff \mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open

$\iff \mathcal{O}_{X,x}$ is reduced $\forall x \in X$

$\iff \mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open aff

(2) $Y \xhookrightarrow{i} X$ $\quad i^*\mathcal{O}_X$ a sheaf on Y

Y: closed subset; X: scheme $\mathcal{O}_X$

Rec: Affine X-Schemes

$\iff \mathcal{O}_X$-algebras q.c.

Closed subschemes of X

$\iff \mathcal{O}_X$-alg. which are quot. of $\mathcal{O}_X$

$\iff$ q.c. ideal sheaves on X

Plan: write down $I \subset \mathcal{O}_X$ q.c. ideal sheaf

s.t. Y_{red} is the corr. closed subsch.

$\underline{\mathrm{Spec}}\ \mathcal{O}_X/I$

Specifically, for $\underset{U=}{\mathrm{Spec}\ A} \underset{\text{open}}{\subset} X$, define $I(U) = \ldots$

Then For general $V \subset X$, $a \in \mathcal{O}_X(V)$ ..

have $a \in \mathcal{I}(V) \overset{\text{def.}}{\iff} a|_U \in \mathcal{I}(U)$

$\forall\, U \subset V$ open aff

$\rightsquigarrow$ need to check that $\mathcal{I} \subset \mathcal{O}_X$ is a sheaf

& does what we want.

$Y \cap \operatorname{Spec} A \xhookrightarrow[\text{closed subset}]{} \operatorname{Spec} A \quad \longleftrightarrow$ radical ideal $I \subset A$

$\mathcal{I}(\operatorname{Spec} A) = I$

(so that $Y = \operatorname{Spec} A/_I \hookrightarrow \operatorname{Spec} A$)

& reduced

In order to exhibit q.c. ideal sheaf $\mathcal{I}$, need to check: - $\operatorname{Spec} A \subset \operatorname{Spec} B \subset X$

$$\begin{array}{ccc} \mathcal{I}(B) & \hookrightarrow & \mathcal{O}_X(B) \\ \downarrow & & \downarrow \\ \mathcal{I}(A) & \hookrightarrow & \mathcal{O}_X(A) \end{array}$$

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quasi-coherent sheaves:

A ring, $M \in A\text{-Mod}$

build $\tilde{M} \in \mathcal{O}_{\operatorname{Spec} A}\text{-Mod}$

$\tilde{M}(D(f)) = M_f$ an A_f-module

main point: functor $M \longmapsto \tilde{M}$

$A\text{-mod} \longrightarrow \mathcal{O}_{\operatorname{Spec} A}\text{-Mod}$

is fully faithful i.e.:

$\operatorname{Hom}_{A\text{-mod}}(M, N) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{O}_{\operatorname{Spec} A}\text{-Mod}}(\tilde{M}, \tilde{N})$.

i.e. : $\mathcal{O}_{\operatorname{Spec} A}$-mod is bigger than A-mod, but an embedding

- X a scheme

$M \in \mathcal{O}_X$-mod is called quasi-coh. if $\forall \operatorname{Spec} A \underset{\text{open}}{\subset} X$,

$M|_{\operatorname{Spec} A} \in \mathcal{O}_{\operatorname{Spec} A}$-Mod is q.c. in prev. sense.

e.g. $\mathcal{O}_X$

- $I(A_f) = I(A)_f$

Explicitly:

$$\begin{array}{ccc} Y \overset{\text{closed}}{\subset} \operatorname{Spec} A & \leftarrow & I \\ \alpha \uparrow & & \\ \alpha^{-1}(Y) \subset \operatorname{Spec} B & & \sqrt{IB} \end{array} \qquad \begin{array}{c} A \\ \downarrow \\ B \end{array}$$

Helpful?

$$\begin{array}{ccc} I & \longrightarrow & A \\ \vdots & & \downarrow \\ \sqrt{IB} & \longrightarrow & B \end{array} \quad (\text{cond } 1 \checkmark)$$

$$\begin{array}{c} I_f \\ \parallel \\ \sqrt{I \cdot A_f} \end{array} \qquad \text{cond}^n\ 2 \quad \checkmark$$

$\left(\frac{a}{f}\right)^n \in I \cdot A_f$

$$\Leftrightarrow f^m \cdot \left(\frac{a}{f}\right)^n \in I$$

$$\Rightarrow (f^? a)^n \in I$$

$$\Rightarrow f^? a \in I$$

$$\Rightarrow \frac{a}{f} \in A_f I$$

$\alpha^\#: A \longrightarrow B \quad V(I) = Y \subset \operatorname{Spec} A$

$\alpha: \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

$\underset{\operatorname{Spec} B}{\overset{\cap}{Q}} \in \alpha^{-1}(Y) \Leftrightarrow \alpha(Q) \in Y \Leftrightarrow (\alpha^\#)^{-1}(Q) \supset I$

$\Leftrightarrow Q \supset \alpha^\#(I) \quad \Leftrightarrow Q \supset \alpha^\#(I) \cdot B$

$\Leftrightarrow Q \in V(IB)$

<u>Summary</u> Found $I \subset \mathcal{O}_X$ q.c. ideal sheaf

s.t. $I(\operatorname{Spec} A) = I(Y \cap \operatorname{Spec} A)$

= ideal defining the reduced closed subsch. of Spec A

$\rightsquigarrow$ obtain $Y_{red} := \underline{\operatorname{Spec}}\, \mathcal{O}_X/I$ some closed subsch. of X

s.t. $Y_{red} \cap \operatorname{Spec} A$ = reduced closed subsch. corr. to $Y \cap \operatorname{Spec} A$

$\therefore$ Y_{red} is a reduced closed subsch. with $|Y_{red}| = Y$.

(3) $\forall$ Z reduced, $\alpha : Z \to X$

$\alpha(|Z|) \subset Y$, then

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, dashed, "\exists!"'] & X \\ & Y_{red} \arrow[u, hook] \end{tikzcd}$$

problem is local on Z & X

$\therefore$ WMA $Z = \operatorname{Spec} B$ $X = \operatorname{Spec} A$ $Y_{red} = \operatorname{Spec} A/I$ ← radical ideal

reduced ring →

$$\begin{tikzcd} B & A \arrow[l, "\alpha^\#"'] \arrow[d, two heads] \\ & A/I \arrow[ul, dashed, "\exists!"] \end{tikzcd}$$

surj. $\therefore$ at most 1 factorization

Need: $\alpha^\#(I) = 0$. $\alpha^{-1}(Y) = \operatorname{Spec} B$

$IB = 0$ $I(\alpha^{-1}(Y)) = \sqrt{IB} = \sqrt{(0)}$

Learned: $\sqrt{IB} = \sqrt{0}$

$IB \subset$ $= (0)$ s/c B reduced $\square$

3. $X \subset \mathbb{A}^n_k$ closed, k alg. closed, X proper.

Show: $|X|$ finite.

i.e. $X \to \operatorname{Spec} k$ ← a proper morr.

Fact

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\text{proper}"'] & Y \arrow[dl, "\text{sep.}"] \\ * & \end{tikzcd}$$

X: any var. at all; Y: any sep. var.

Then f is proper.

In part.: $f(X) \subset Y$ a closed subset.

n=1 $X \subset \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$

closed subset.

i.e. $\mathbb{A}^1$ or finite subset

And: $\overline{X} = X \subset \mathbb{P}^1$

but $\overline{\mathbb{A}^1} = \mathbb{P}^2$ ∴ $\mathbb{A}^1$ not proper

∴ X = a finite coll. of points

n>1:

$$\begin{tikzcd} X \arrow[r, hook] & \mathbb{A}^n \arrow[d, "p_i"] \\ p_i(X) \arrow[r, hook] & \mathbb{A}^1 \end{tikzcd}$$

$p_i(X)$ closed

i.e.: the i-th coord. of points in X takes only finitely many diff. values

∴ $|X|$ finite.

Deduction

$$\begin{tikzcd} X \arrow[d, "\text{proper + affine}"] & X_s \arrow[l] \arrow[d] & X_{\bar{s}} \arrow[l] \arrow[d, "\text{aff. + proper}"] \\ S & \operatorname{Spec} k(s) \arrow[l] & \operatorname{Spec} k(\bar{s}) \arrow[l] \end{tikzcd}$$

$s \in S$

∴ $X \to S$ is "quasi-finite"