

1. (1) $X \in \text{Sch.}$ is reduced

$\Leftrightarrow \mathcal{O}_X$ has no nilpotents

$\Leftrightarrow \mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open

$\Leftrightarrow \mathcal{O}_{X,x}$ is reduced $\forall x \in X$

$\Leftrightarrow \mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open aff

(2) $Y \xrightarrow{i} X$ $i^* \mathcal{O}_X$ a sheaf on Y

\uparrow closed subset
 \uparrow sheaf \mathcal{O}_X

Rec: Affine X -Scheme g.c.

$\Leftrightarrow \mathcal{O}_X$ -algebras

closed subsets of X

$\Leftrightarrow \mathcal{O}_X$ -id. which are g.c.

\Leftrightarrow g.c. ideal sheaves on X

Plan: work down $I \subset \mathcal{O}_X$ g.c. ideal sheaf

s.t. Y_{red} is the corr. closed subset.

Spec \mathcal{O}_X/I

Specifically, for $\text{Spec } A \subset X$, define $I(U) = \dots$

Then For general $V \subset X$, $\mathcal{O}_X(V) \dots$

$$\text{have } a \in I(U) \stackrel{\text{def.}}{\iff} a|_U \in I(U)$$

\implies need to check that $I \subset \mathcal{O}_X$ is a sheaf $\uparrow U \subset V$ open aff
& does what we want.

$$Y \cap \text{Spec } A \hookrightarrow \text{Spec } A \quad \longleftrightarrow \quad \text{radical ideal } I \subset A$$

closed
subset

$$I(\text{Spec } A) = \underline{I}$$

(so that $Y = \text{Spec } A_{\underline{I}} \hookrightarrow \text{Spec } A$)
& reduced

In order to exhibit g.c. ideal sheaf \underline{I} , need
to check: $\text{Spec } A \subset \text{Spec } B \subset X \quad \begin{matrix} I(B) \hookrightarrow \mathcal{O}_X(B) \\ \downarrow \\ I(A) \hookrightarrow \mathcal{O}_X(A) \end{matrix}$

quasi-coherent sheaves:

A ring, $M \in A\text{-Mod}$

build $\tilde{M} \in \mathcal{O}_{\text{Spec } A}\text{-Mod}$

$$\tilde{M}(D(f)) = M_f \text{ can } A_f\text{-module}$$

main point: functor $M \mapsto \tilde{M}$

$A\text{-mod} \rightarrow \mathcal{O}_{\text{Spec } A}\text{-Mod}$
is fully faithful i.e.:

$$\text{Hom}_{A\text{-mod}}(M, N) \cong \text{Hom}_{\mathcal{O}_{\text{Spec } A}\text{-Mod}}(\tilde{M}, \tilde{N})$$

i.e. : $\mathcal{O}_{\text{Spec } A}$ -mod is bigger than A -mod, Set an embedding

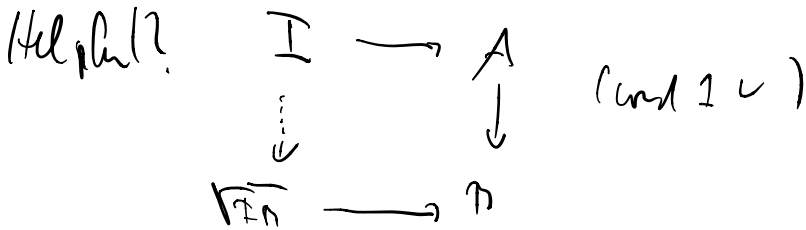
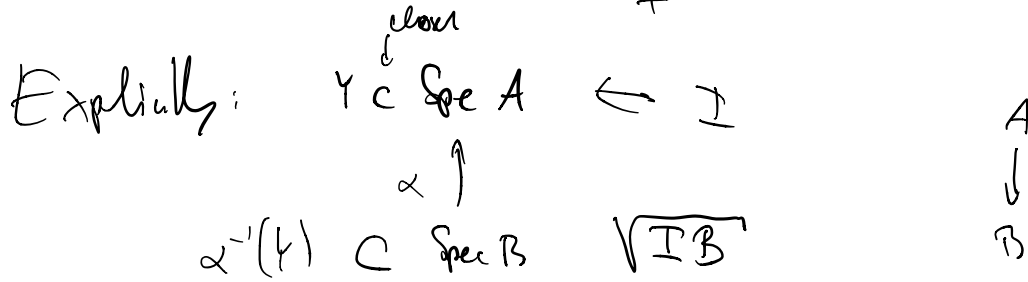
- X a scheme
 $M \in \mathcal{O}_X$ -mod is called quasi-coh. if $\forall \text{Spec } A \subset X$

$$M|_{\text{Spec } A} \in \mathcal{O}_{\text{Spec } A}$$

is q.c. in prev. sense.

e.g. \mathcal{O}_X

$$- I(A_f) = I(A)_f$$



$$\begin{array}{ccc}
 I_f & & \\
 \parallel & \text{Cond } 2 & \checkmark \\
 \sqrt{I \cdot A_f} & & \left(\frac{a}{f}\right)^n \in I \cdot A_f
 \end{array}$$

$$\Leftrightarrow f^m \cdot \begin{pmatrix} a \\ f \end{pmatrix}^m \in I$$

$$\Rightarrow (f^m \cdot a)^m \in I$$

$$\Rightarrow f^m \cdot a \in I$$

$$\Rightarrow \frac{a}{f} \in A_f I$$

$$\alpha^\# : A \rightarrow B \quad V(I) = Y \subset \text{Spec } A$$

$$\alpha : \text{Spec } B \rightarrow \text{Spec } A$$

$$Q \in \alpha^{-1}(Y) \Leftrightarrow \alpha(Q) \in Y \Leftrightarrow (\alpha^\#)^{-1}(Q) \supset I$$

$$\uparrow$$

$$\text{Spec } B \quad \Leftrightarrow Q \supset \alpha^\#(I) \Leftrightarrow Q \supset \alpha^\#(I) \cdot B$$

$$\Leftrightarrow Q \in V(I_B)$$

Summary For an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$

$$\text{s.t. } \mathcal{I}(\text{Spec } A) = I(Y \cap \text{Spec } A)$$

$$= \text{ideal defining the reduced closed subsch. of Spec } A$$

\leadsto define $Y_{\text{red}} := \underline{\text{Spec}} \frac{\mathcal{O}_X}{\mathcal{I}}$ same closed subsch. of X

$$\text{s.t. } Y_{\text{red}} \cap \text{Spec } A = \text{reduced closed subsch. corr. to } I \cap \text{Spec } A$$

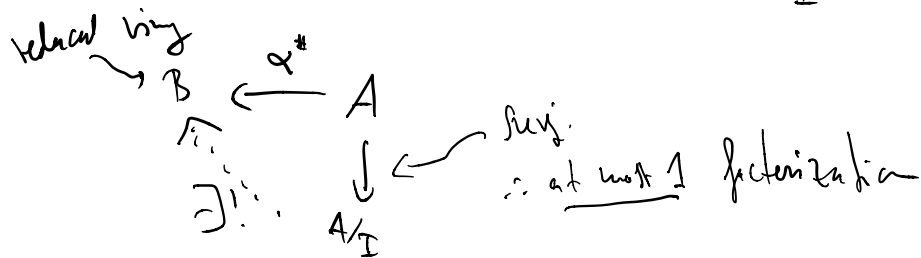
$\therefore Y_{\text{red}}$ is a reduced closed subsch. with $|Y_{\text{red}}| = Y$.

(3) $\forall Z$ reduced, $\alpha: Z \rightarrow X$

$$\alpha(Z) \subset Y, \text{ then } \begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & \searrow & \downarrow \\ \exists! & \longrightarrow & Y_{\text{red}} \end{array}$$

problem is local on Z & X

\therefore WMA $Z = \text{Spec } B$ $X = \text{Spec } A$ $Y_{\text{red}} = \text{Spec } \frac{A}{\sqrt{0}}$ radical ideal



Need: $\alpha^*(\sqrt{0}) = 0$

$\alpha^{-1}(Y) = \text{Spec } B$

$\sqrt{0} = 0$

$I(\alpha^{-1}(Y)) = \sqrt{IB} = \sqrt{(0)}$

Lemma: $\sqrt{\sqrt{IB}} = \sqrt{0}$

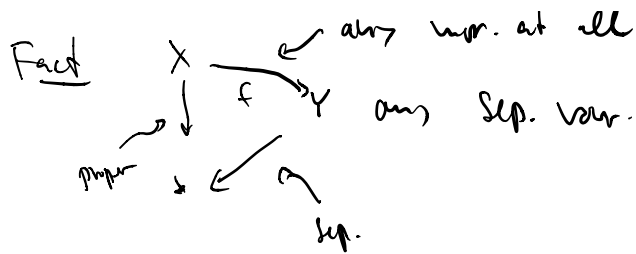
$\sqrt{IB} = (0)$ since B reduced



3. $X \subset \mathbb{A}_k^n$, k alg. closed, X proper.
closed

Show: $|X|$ finite.

i.e. $\begin{array}{c} X \\ \downarrow \\ \text{Spec } k \end{array}$ a proper morphism.



Then f is proper.

In part: $f(X) \subset Y$
a closed subset.

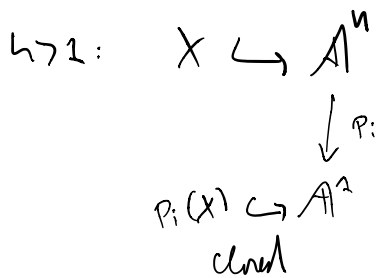
$n=1$ $X \subset \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$
closed subset.

i.e. \mathbb{A}^1 or finite subset

And: $\overline{X} = X \subset \mathbb{P}^1$

but $\overline{\mathbb{A}^1} = \mathbb{P}^1 \therefore \mathbb{A}^1$ not proper

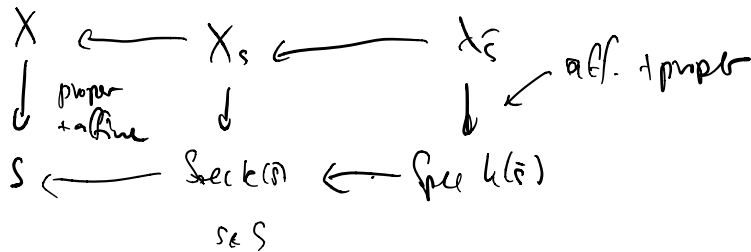
$\therefore X$ is a finite coll. of points



i.e.: the i -th coord. of points in X
takes only finitely many diff. values

$\therefore |X|$ finite.

Deduction



$\therefore X$ is "quasi-finite"