1. $S_\cdot$ graded ring $\mathbb{N}$

$M_\cdot$ graded $S_\cdot$-module $\mathbb{Z}$

Construct: $\widetilde{M_\cdot} \in QCoh(\operatorname{Proj} S_\cdot)$

s.t. $\widetilde{M_\cdot}\big|_{D_+(f)} \simeq \widetilde{M_{(f)}} \in QCoh(D_+(f))$

$\simeq S_{(f)}\text{-Mod}$

graded $S_\cdot$-module : $M = \bigoplus_{i \in \mathbb{Z}} M_i$ ab. gp.

S-module

$S_i \cdot M_j \subset M_{i+j}$ (in part. $S_0 \cdot M_i \subset M_i$ $\rightsquigarrow$ M_i is an S_0-mod.)

Constructing $\widetilde{M_\cdot}$

write it down "directly"

$F(U) = \{ \ldots \}$

"cocycle condition"

$F_\alpha|_{U_\alpha \cap U_\beta \cap U_\gamma} \xrightarrow{f_{\alpha\beta}} F_\beta|_{U_\alpha \cap U_\beta \cap U_\gamma}$

gluing

X a scheme

$\{U_\alpha\}$ open cover

$F_\alpha \in Shv(U_\alpha)$

QCoh ...

$f_{\alpha\beta} : F_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\simeq} F_\beta|_{U_\alpha \cap U_\beta}$

$$f_{\alpha\gamma} \circlearrowleft \quad \downarrow f_{\beta\gamma} \qquad \forall \alpha, \beta, \gamma$$

$$F_\gamma|_{U_\alpha \cap U_\beta \cap U_\gamma}$$

$$\tilde{M}(U) = \left\{ \begin{matrix} \coprod_p M_{(p)} \\ \uparrow \sigma \\ U \end{matrix} \;\middle|\; \text{locally } \sigma \text{ is of the form } \frac{m}{f} \quad \begin{matrix} m \in M_i \\ f \in S_i \end{matrix} \right\}$$

make into $\mathcal{O}_{\mathrm{Proj}(S)}$-module $\rightsquigarrow$ clear.

Have $\tilde{M}(D_+(f)) \longleftarrow M_{(f)}$

$$\frac{m}{f^i} \longleftarrow\!\shortmid \frac{m}{f^i} \qquad \begin{matrix} m \in M_{i \cdot n} \\ f \in S_n \end{matrix} .$$

$\rightsquigarrow$ yields hom. of sheaves on $\operatorname{Spec} S_{(f)} \cong D_+(f)$.

$\rightsquigarrow$ check iso on stalks

stalks of $\tilde{M}|_{D_+(f)}$: given by $M_{(p)}$

stalks of $\widetilde{M_{(f)}}$: — " — $\square$

<u>2nd approach</u> Cover Proj S. by $\{D_+(f) \cong \operatorname{Spec} S_{(f)}\}$

f any homog. elt.

$\mathrm{QCoh}(D_+(f)) \simeq S_{(f)}$-mod

$F_f = \widetilde{M_{(f)}}$

ASSUME $S_\cdot$ gen in deg 1 $f, g, \ldots \in S_1$

Need $\alpha_{fg} : \widetilde{M_{(f)}}\big|_{D_+(fg)} \xrightarrow{\cong} \widetilde{M_{(g)}}\big|_{D_+(fg)}$

$QCoh(D_+(fg)) = S_{(fg)}\text{-Mod}$

$$\left(M_{(f)}\right)_{\frac{g}{f}} \xrightarrow{\cong} \left(M_{(g)}\right)_{\frac{f}{g}}$$

both are can. $\cong M_{(fg)}$

Cocycle cond^n:

$$\begin{tikzcd}
\left(\left(M_{(f)}\right)_{\frac{g}{f}}\right)_{\frac{h}{f}} \arrow[r, "\cong"] \arrow[d, "\cong"] & \left(\left(M_{(g)}\right)_{\frac{f}{g}}\right)_{\frac{h}{g}} \arrow[dl, "\cong"] \\
\left(\left(M_{(h)}\right)_{\frac{f}{h}}\right)_{\frac{g}{h}} &
\end{tikzcd}$$

all 3 are can. iso to $M_{(fgh)}$

One more idea:

$$\begin{tikzcd}
A^{(J)} \arrow[r] & A^{(I)} \arrow[r, two heads] & M \\
 & J \arrow[ul, no head] &
\end{tikzcd}$$

In graded sit^n:

$$\bigoplus_{i\in\mathbb{Z}} S_{\cdot+i}^{(J_i)} \longrightarrow \bigoplus_{i\in\mathbb{Z}} S_{\cdot+i}^{(I_i)} \twoheadrightarrow M$$

Can build $\widetilde{M}$ if you have $\widetilde{S_{\cdot+i}}$.

$i=0$: $\mathcal{O}_{\mathrm{Proj}(S)}$

$i\neq 0$: ?? $\quad \widetilde{S_{\cdot+i}} =: \mathcal{O}(i)$

very important sheaves on $\mathrm{Proj}\, S_\cdot$ "twisting sheaves"

2. $\mathrm{Sch} \hookrightarrow P(\mathrm{Sch}) = \mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set})$

$X \longmapsto R_X$

$R_X(Y) = \mathrm{Hom}_{\mathrm{Sch}}(Y, X)$

Fact: $R_{\mathbb{P}^n_{\mathbb{Z}}}$ has an interesting interpretation.

Define $F \in P(\mathrm{Sch})$

$$F(X) = \left\{ (\mathcal{E}, \alpha) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ an invertible sheaf} \\ \alpha : \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{E} \end{array} \right\} \Big/ \cong$$

i.e. $n+1$ elts of $\Gamma(X, \mathcal{E})$

$$\begin{tikzcd} & \mathcal{O}_X^{n+1} \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathcal{E} \arrow[rr, "\cong"] & & \mathcal{E}' \end{tikzcd}$$

Shall show: $F \cong R_{\mathbb{P}^n_{\mathbb{Z}}}$.

(1) $X \in Sch$

$\mathcal{L}$ invertible $\mathcal{O}_X$-mod.

$\sigma \in \Gamma(X, \mathcal{L})$

$D(\sigma) = \{ x \in X \mid \mathcal{L}_x$ is gen. as an $\mathcal{O}_{X,x}$-mod. by $\sigma_x \in \mathcal{L}_x \}$

May reduce to affine case.

$X = \operatorname{Spec} A$

$\mathcal{L} = \widetilde{M}$ $\quad M$ some inv. A-mod

$\sigma \in M$

Locally on X, $\mathcal{L}_X \cong \mathcal{O}_X$

$\sigma \longleftrightarrow f$

Iso is not canonical $\rightsquigarrow$ f is well-defined up to mult^n by a unit

CLAIM: $D(\sigma) = D(f) = D(\underset{\text{unit}}{u} \cdot f)$

i.e.: Give $f \in A$

RTP $D(f) = \{ \mathfrak{p} \mid \underbrace{A_{\mathfrak{p}} \text{ is gen by } f \text{ as an } A_{\mathfrak{p}}\text{-mod}} \}$

$\Leftrightarrow f \in A_{\mathfrak{p}}^{\times}$

$\Leftrightarrow f \notin \mathfrak{p}$

Show: if $(\sigma_0, \ldots, \sigma_n) : \mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L}$

then $X = \bigcup_i D(\sigma_i)$

Informally: $D(\sigma_i) = \{ x \mid \sigma_i(x) \neq 0 \}$

Suppose not a cover. I.e. $\exists x \in X$ s.t. $\sigma_i(x) = 0$ $\forall i$.

If $\mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{L}$ then <u>any</u> local section σ of $\mathcal{L}$ would vanish at x.

i.e. $\mathcal{L}_x = 0$ $\times$

$\cong$!

$\mathcal{O}_x \neq 0$

More formally: Check locally. WMA $\mathcal{L} \cong \mathcal{O}$,

$X = \operatorname{Spec} A$

Problem Says: if $f_1, \ldots, f_n \in A$ gen. A as a module,

then $D(f_1) \cup \ldots \cup D(f_n) = \operatorname{Spec} A$.

Know this! [Spec $A \setminus (D(f_1) \cup \dots \cup D(f_n))$
$= V(f_1, \dots, f_n)$
$= \emptyset \iff (f_1, \dots, f_n) = A$]

(2) $\mathcal{O}(n) = \widetilde{S_{\bullet+n}}$ on $\mathrm{Proj}(S)$ ← gen. in deg. 1

Case $S = \mathbb{Z}[T_0, \dots, T_n]$
$\mathrm{Proj}(S_\bullet) = \mathbb{P}^n_{\mathbb{Z}}$
Construct. $\mathcal{O}^{n+1} \twoheadrightarrow \mathcal{O}(1)$.

Show $\mathcal{O}(1)$ invertible.

loc. free rk 1, i.e. loc. $\cong \mathcal{O}_X$

$\exists\ \mathcal{O}(1)^{-1}$ s.t. $\mathcal{O}(1)^{-1} \otimes \mathcal{O}(1) \cong \mathcal{O}_X$

$\mathcal{O}(1)|_{D_+(f)} \cong \widetilde{S_{\bullet+1(f)}}$
$f = T_i$

Want to show: $(S_{\bullet+1})_{(f)} \cong (S_\bullet)_{(f)}$

$\left\{ \frac{x}{f^n} \;\middle|\; \deg x = n+1 \right\}$
$|f| = 1$

S_f ← graded module

$(S_f)_0 = S_{(f)}$

Graded $S_\bullet$-modules $\to$ QCoh(Proj)
is symm. monoid.

$S_{\bullet+1} \otimes_{S_\bullet} S_{\bullet-1} \xrightarrow{\sim} S_\bullet$

$a \otimes s \longmapsto a \cdot s$

& $S \otimes_S S \cong S$ via mult^n

$(S_f)_i = (S_{\cdot+1})_{(f)}$

Claim: $(S_f)_i$ are all iso!

Indeed mult' by f is iso from $(S_f)_i$ to $(S_f)_{i+1}$.