

1. X affine scheme

$0 \rightarrow F_1 \rightarrow F_2 \xrightarrow{\sigma} F_3 \rightarrow 0$
 exact seq. of \mathcal{O}_X -modules. Assume that F_2 is \mathcal{O}_X -loc. free.

Show that $0 \rightarrow F_1(X) \rightarrow F_2(X) \rightarrow F_3(X) \rightarrow 0$
 is exact. $H^i(X, -)$

Digression: $\Gamma(X, -)$: (Abelian sheaves on X) \rightarrow (ab. grps)
 is left exact. \rightsquigarrow right derived functors
 denoted $H^i(X, -)$.

\rightsquigarrow LES $0 \rightarrow F_1(X) \rightarrow F_2(X) \rightarrow F_3(X) \rightarrow 0$
 $\rightarrow H^1(X, F_1) \rightarrow H^1(X, F_2) \rightarrow \dots$
 $= 0?$

YES - later in course.

Digression: exactness of seq. of sheaves

1st answer: $F \rightarrow G \rightarrow H$ is exact at G if
 $k \in X$, $F_x \rightarrow G_x \rightarrow H_x$ is exact.
 GAS

2nd answer: $F \rightarrow G$ is injective (i.e. $0 \rightarrow F \rightarrow G$ is exact)

$$\Leftrightarrow F(U) \hookrightarrow G(U) \quad \forall U$$

$F \xrightarrow{f} G$ is surjective if $\forall \alpha \in X$
 $\forall a \in G(U) \exists$ covering U_α of U & elts.
 $b_\alpha \in F(U_\alpha)$ st. $f(b_\alpha) = a|_{U_\alpha} \quad \forall \alpha$.

$F \rightarrow G \xrightarrow{f} H$ is exact if $F \xrightarrow{f} G \rightarrow H$
 [ker $f \hookrightarrow G$ can be computed sectionwise]

For the hint:

$a \in F_3(X)$	$f \in \mathcal{O}(X)$
$b \in F_2(D(f))$	$f(b) = a _{D(f)}$

Want: $\exists n \gg 0$
 $b' \in F_2(X)$ st. $f(b') = f^n \cdot a$.

$\exists X = \bigcup_{i=1}^n D(g_i)$ & $s_i \in F_2(D(g_i))$ st. $f(s_i) = a|_{D(g_i)}$.

$$f \left(\underbrace{s_i|_{D(g_i) \cap D(f)} - s_j|_{D(g_i) \cap D(f)}} \right) = 0$$

$$\therefore \in F_1(\underbrace{D(g_i) \cap D(f)})$$

$$\mathcal{D}(f, g) \subset \mathcal{D}(g_i)$$

M sect. over $\mathcal{D}(g_i)$

$S^1 M$ sect. over $\mathcal{D}(g_i)$

$S^1 \{f^m \mid m \geq 0\}$

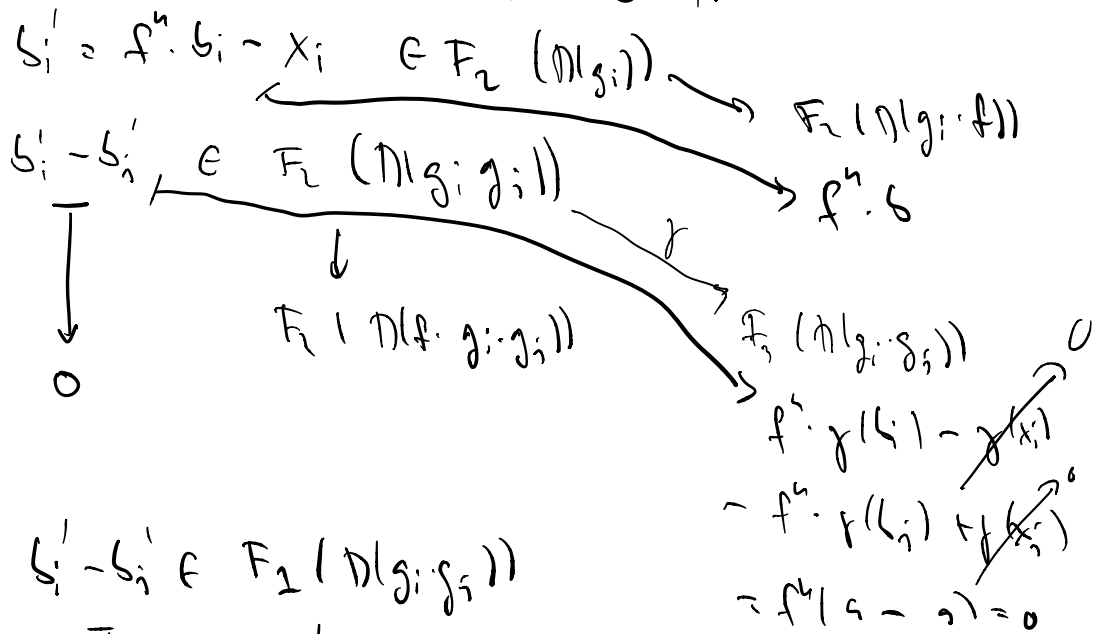
LA If $\exists g_i$, $x \in F_2(\mathcal{D}(g_i))$

then $g^m x$ extends to $F(X)$ for $m \gg 0$.

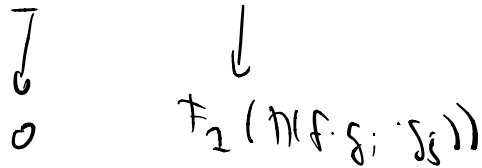
Apply to $X = \mathcal{D}(g_i) \rightsquigarrow (b_i - b) \cdot f^m$ extends to $\mathcal{D}(g_i)$
for m big enough.

with $\delta_i \in F_2(\mathcal{D}(g_i))$ for this extension.

WMA m works for all i .



Hence: $b_i' - b_i' \in F_2(\mathcal{D}(g_i; g_i))$



$\ker \mu \rightarrow \mu_f$
= $\{m \in M \mid f^m = 0 \text{ for } m \geq 0\}$

$\therefore f^p \cdot (b_i' - b_i') = 0$ for $p \gg 0$.

WMA since p works for all i, j .

$$s_i'' = f^p \cdot s_i' \in F_2(\mathcal{O}(g_i))$$

$$s_i'' - s_i'' = 0 \in F_2(\mathcal{O}(g_i)) \Rightarrow \exists s'' \in F_2(X)$$

$$f(s_i'') = f^{4p} \cdot a \quad s''|_{\mathcal{O}(g_i)} = s_i''$$

$$\Rightarrow f(s'') = f^{4p} \cdot a$$

□

$$0 \rightarrow F_1 \rightarrow F_2 \xrightarrow{\gamma} F_3 \rightarrow 0$$

\uparrow
p
z.cob.

$$a \in F_3(X)$$

$\exists f_1, \dots, f_n \in \mathcal{O}(X)$ s.t. can lift a
over $D(f_i)$. $\cup_i D(f_i) = X$. ⊗

Hint $\rightsquigarrow \exists s_i \in F_2(\mathcal{O}(f_i))$ s.t. $f(s_i) = f_i^N \cdot a$

Can lift $(\sum_i c_i f_i^N) \cdot a$ for any $c_i \in \mathcal{O}(X)$
($\cup_i \mathcal{O}(f_i)$).

$$\otimes \Rightarrow (f_1, \dots, f_n) = (1)$$

$$\Rightarrow (f_1^N, \dots, f_n^N) = (1) \quad \therefore \exists c_i \text{ s.t. } \sum c_i f_i^N = 1$$

\therefore can lift a □

2. X she.

Show that γ cob. sheaves are closed

under ker , image, cok , extensions.

$$\begin{array}{l} \text{i.e. } 0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0 \\ \text{SES of } \mathcal{O}_X\text{-mod with } F_1, F_2 \text{ g.coh.} \\ \Rightarrow F_2 \text{ is g.coh.} \end{array}$$

We may assume that $X = \text{Spec } A$ is affine.

(Hc being g.coh. is local.)

$A\text{-mod} \longrightarrow \text{Shv. of } \mathcal{O}_X\text{-mod.}$

$$M \longmapsto \tilde{M}$$

fully faithful functor with ex. image the g.coh. sheaves.

Need to show: if $0 \rightarrow M_1 \rightarrow M_2 \xrightarrow{f} M_3$ is exact

then $0 \rightarrow \tilde{M}_1 \rightarrow \tilde{M}_2 \rightarrow \tilde{M}_3$ is exact

(for ker).

Check on stalks: $P \in \text{Spec } A$

$$\begin{array}{ccccccc} 0 & \rightarrow & (\tilde{M}_1)_P & \rightarrow & (\tilde{M}_2)_P & \rightarrow & (\tilde{M}_3)_P \text{ exact?} \\ & & \parallel & & \parallel & & \parallel \\ & & (M_1)_P & & (M_2)_P & & (M_3)_P \end{array}$$

Yes, h.c. loc^s is an exact functor.

Extensions: $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$

exact seq. of \mathcal{O}_X -mod, $X = \text{Spec } A$,

F_1, F_3 x.coh.

Show F_2 q.coh.

$$\begin{array}{ccc}
 M & \xrightarrow{\quad} & N \\
 \downarrow & \xrightarrow{\quad} & \downarrow \\
 A\text{-mod} & \xrightarrow{\quad} & \mathcal{O}_X\text{-Mod} \\
 \\
 F(X) & \xleftarrow{\quad} & F
 \end{array}$$

May show these are adjoint.

In particular: There is a functorial morphism

$$\widetilde{F(X)} \xrightarrow{c_F} F$$

F is q.coh iff c_F is iso.

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0 \quad \text{Exact}$$

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \widetilde{F_1} & & \widetilde{F_2} & & \widetilde{F_3} \\
 0 & \rightarrow & \widetilde{F_1(X)} & \rightarrow & \widetilde{F_2(X)} & \rightarrow & \widetilde{F_3(X)} \rightarrow 0
 \end{array}$$

exact by ex. (1)

iso by 5-lemma.