1. X affine scheme

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \xrightarrow{\sigma} \mathcal{F}_3 \to 0$

exact seq. of sheaves of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}_1$ is q.coh.

Show that $0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to 0$

is exact.

Digression: $\underset{H^0(X,-)}{\overset{\|}{\Gamma}}(X,-) : (\text{Abelian sheaves on } X) \to (\text{ab. grps})$

is left exact. $\rightsquigarrow$ right derived functors

denoted $H^i(X,-)$.

$\rightsquigarrow$ LES $0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to$

$\to \underbrace{H^1(X,\mathcal{F}_1)}_{=0?} \to H^1(X,\mathcal{F}_2) \to \cdots$

YES - later in course.

Digression: exactness of seq. of sheaves

1st answer: $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact at $\mathcal{G}$ iff

$\forall x \in X$, $\mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x$ is exact.

2nd answer: $\mathcal{F} \to \mathcal{G}$ is injective (i.e. $0 \to \mathcal{F} \to \mathcal{G}$ is exact)

$\Leftrightarrow F(U) \hookrightarrow G(U) \ \forall U$

$F \xrightarrow{f} G$ is surjective if $\forall U \subset X$
$\forall a \in G(U)$ $\exists$ covering V_α of U & elts.
$s_\alpha \in F(V_\alpha)$ st. $f(s_\alpha) = a|_{V_\alpha}$ $\forall \alpha$.

$F \to G \xrightarrow{f} H$ is exact if $F \twoheadrightarrow \ker f$
[$\ker f \hookrightarrow G$ can be computed section-wise]

For the hint: $a \in F_3(X)$ $\quad f \in \mathcal{O}(X)$
$s \in F_2(D(f))$ $\quad g(s) = a|_{D(f)}$.
Want: $\exists n \gg 0$
$s' \in F_2(X)$ s.t. $g(s') = f^n \cdot a$.

$\exists X = \bigcup_{i=1}^{n} D(g_i)$ & $s_i \in F_2(D(g_i))$ s.t. $g(s_i) = a|_{D(g_i)}$.

$$g\Big(\underbrace{s_i|_{D(g_i) \cap D(f)} - s|_{D(g_i) \cap D(f)}}\Big) = 0$$

$\therefore \in F_1(\underbrace{D(g_i) \cap D(f)})$

$D(f \cdot g_i) \subset D(g_i)$ $\qquad$ M sect. over $D(g_i)$

$S^{-1}M$ $\qquad$ sect. over $D(f \cdot g_i)$

$S = \{ f^m \mid m \geq 0 \}$

LA If $\mathcal{F}$ q.coh., $x \in \mathcal{F}_1(D(g))$

then $g^n x$ extends to $\mathcal{F}(X)$ for $n \gg 0$.

Apply to $X = D(g_i) \rightsquigarrow (s_i - s) \cdot f^n$ extends to $D(g_i)$

for n big enough.

With $\delta_i \in \mathcal{F}_1(D(g_i))$ for this extension.

WMA n works for all i.

$$\begin{tikzcd}
s_i' = f^n \cdot s_i - x_i \in \mathcal{F}_2(D(g_i)) \arrow[r] & \mathcal{F}_2(D(g_i \cdot f)) \\
s_i' - s_j' \in \mathcal{F}_2(D(g_i g_j)) \arrow[d] \arrow[r, "\gamma"] & \mathcal{F}_3(D(g_i \cdot g_j)) \\
\mathcal{F}_2(D(f \cdot g_i \cdot g_j)) &
\end{tikzcd}$$

$s_i' \mapsto f^n \cdot s$, $\quad s_i' - s_j' \mapsto 0$

$f^n \cdot \gamma(s_i) - \cancel{\gamma(x_i)}^{\,0}$

$- f^n \cdot \gamma(s_j) + \cancel{\gamma(x_j)}^{\,0}$

$= f^n(s - s) = 0$

Hence: $s_i' - s_j' \in \mathcal{F}_1(D(g_i \cdot g_j))$

$$\begin{tikzcd}
s_i' - s_j' \arrow[d, mapsto] & \mathcal{F}_1(D(g_i \cdot g_j)) \arrow[d] \\
0 & \mathcal{F}_2(D(f \cdot g_i \cdot g_j))
\end{tikzcd}$$

$\ker M \longrightarrow M_f$
$= \{ m \in M \mid f^n m = 0 \text{ for some } n \}$

$\therefore f^p \cdot (s_i' - s_j') = 0$ for $p \gg 0$.

WMA some p works for all i, j.

$s_i'' = f^p \cdot s_i' \in \mathcal{F}_2(D(g_i))$

$s_i'' - s_j'' = 0 \in \mathcal{F}_2(D(g_i \cdot g_j)) \Rightarrow \exists s'' \in \mathcal{F}_2(X)$

$\gamma(s_i'') = f^{n+p} \cdot a$ $\qquad s''|_{D(g_i)} = s_i''$

$\Rightarrow \gamma(s'') = f^{n+p} \cdot a$

$\square$

$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \xrightarrow{\gamma} \mathcal{F}_3 \to 0$ $\qquad a \in \mathcal{F}_3(X)$

q.coh.

$\exists f_1, \dots, f_n \in \mathcal{O}(X)$ s.t. can lift a over $D(f_i)$. $\bigcup_i D(f_i) = X$. $\circledast$

Hint $\leadsto$ $\underset{N}{\exists}$ $s_i \in \mathcal{F}_2(D(f_i))$ s.t. $\gamma(s_i) = f_i^N \cdot a$

Can lift $\left(\sum_i c_i f_i^N\right) \cdot a$ for any $c_i \in \mathcal{O}(X)$

(lift $\sum_i c_i \cdot s_i$).

$\circledast \Rightarrow (f_1, \dots, f_n) = (1)$

$\Rightarrow (f_1^N, \dots, f_n^N) = (1)$ $\quad \therefore \exists\, c_i$ s.t. $\sum c_i f_i^N = 1$

$\therefore$ can lift a $\square$

2. X scheme.

Show that q.coh. sheaves on X are closed under ker, image, cok, extensions.

i.e. $0 \to F_1 \to F_2 \to F_3 \to 0$
SES of $\mathcal{O}_X^{sh.}$-mod with F_1, F_3 q.coh.
$\Rightarrow F_2$ is q.coh.

We may assume that $X = \operatorname{Spec} A$ is affine.
(the being q.coh. is local.)

$$\text{A-mod} \longrightarrow \text{Shv. of } \mathcal{O}_X\text{-mod.}$$
$$M \longmapsto \widetilde{M}$$

fully faithful functor with ess. image the q.coh. sheaves.

Need to show: if $0 \to M_1 \to M_2 \xrightarrow{f} M_3$ is exact

then $0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3}$ is exact

(for kernels).

Check on stalks: $p \in \operatorname{Spec} A$

$$\text{Is } 0 \to (\widetilde{M_1})_p \to (\widetilde{M_2})_p \to (\widetilde{M_3})_p \text{ exact?}$$
$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ (M_1)_p & (M_2)_p & (M_3)_p \end{array}$$

Yes, b/c loc^n is an exact functor.

Extensions: $0 \to F_1 \to F_2 \to F_3 \to 0$

exact seq. of $\mathcal{O}_X$-mod, $X = \operatorname{Spec} A$,

F_1, F_3 q.coh.

Show F_2 q.coh.

$$M \longmapsto \tilde{M}$$

$$A\text{-mod} \rightleftarrows \mathcal{O}_X\text{-Mod}$$

$$F(X) \longleftarrow F$$

May show these are adjoint.

In particular: There is a functorial morphism

$$\widetilde{F(X)} \xrightarrow{c_F} F$$

F is q.coh iff c_F is iso.

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & F_3 \arrow[r] & 0 & \text{exact} \\
0 \arrow[r] \arrow[u] & \widetilde{F_1(X)} \arrow[r] \arrow[u, "\simeq"] & \widetilde{F_2(X)} \arrow[r] \arrow[u] & \widetilde{F_3(X)} \arrow[r] \arrow[u, "\simeq"] & 0 \arrow[u] & \text{exact by ex. (1)}
\end{tikzcd}$$

iso by 5-lemma.