1. $g, f : X \to Y \in Sch_S$ $\quad \exists\, U \subset X$ dense open
$f(x) = g(x)$ $\forall x \in U$.

X reduced, $Y \to S$ sep. 1) Show: $f = g$
2) Counterex. w/o ass.

$$
\begin{tikzcd}
X \arrow[r, "(f,g)"] \arrow[rr, bend left, "f"] \arrow[rrr, bend left=50, "g"] & Y \times_S Y \arrow[r] \arrow[r, shift left] & Y \\
X' \arrow[u, hook] \arrow[r] & Y \arrow[u, hook]
\end{tikzcd}
$$

$\leftarrow$ closed imm.

$X' = \{x \in X \mid f(x) = g(x)\}$ $\leftarrow$ closed

$U \subset$ by ass.

$U \subset$ dense

$\therefore \overline{U} \subset X'$

$\overline{U} = X$

$\therefore X' \hookrightarrow X$ is a closed subsch on all of X

Then $X' \xrightarrow{\sim} X$ b/c X <u>reduced</u>.

$\hookrightarrow$ construction

$$
\begin{tikzcd}
X' \arrow[r] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y
\end{tikzcd}
$$

are the same maps

$\therefore$ want to be iso

$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]/\varepsilon^2$

$k[\varepsilon]/\varepsilon^2 \to k$

Counterexamples: 1) X non-reduced

$$X = \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \underset{f}{\overset{g}{\rightrightarrows}} Y = \operatorname{Spec} k[\varepsilon]$$

$\| \quad \{(0)\}$ Then $f(x) = g(x) \quad \forall x \in X$

$$\begin{tikzcd} k[\varepsilon]/\varepsilon^2 & k[\varepsilon]/\varepsilon^2 \arrow[l, bend right, "f^\#"'] \arrow[l, bend left, "g^\#"] \end{tikzcd}$$

k-algebra maps $k[\varepsilon]/\varepsilon^2 \to R$ $\quad R = \{k[x] \xrightarrow[\text{k-alg. map}]{} R\}$

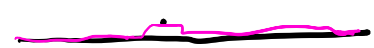
$= \{r \in R \mid r^2 = 0\}$ $\qquad x \longmapsto r \in R$

take $f^\#, g^\#$ corresponding to $0, \varepsilon$

2) Y insep. / S

Y = affine line with origin doubled

$$X = \mathbb{A}^1 \xrightarrow{f} = \mathbb{A}^1 \coprod_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$$

g $=$ 

$U = \mathbb{A}^1 \setminus 0$

2. X sep. $\quad U, V \subset X$ open aff.

Show: $U \cap V \subset X$ is affine.

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & X \arrow[d, "\text{sep.}"] \\ U \times V \arrow[r, hook, "\text{open}"'] & X \times X \end{tikzcd}$$

$U \times V$ affine

closed immersion

open imm.

$X' = U \cap V$

closed subsch. of $U \times V$ which is affine.

3. X/k sep. + integral, f.t./k $\qquad$ $X \to \operatorname{Spec} k$ sep. $\Leftrightarrow X \to \operatorname{Spec} \mathbb{Z}$ sep.

$K = k(X)$ $\quad$ $R \subset K$ val^n ring

Show: there is at most 1 pt $x \in X$ s.t. $\mathcal{O}_{X,x} \subset R$ $+ \; m_x \subset m_R$

$\cap$ K

Valuative criterion of separatedness

f sep. $\Leftrightarrow$ $\forall$ val^n ring R with frac. field K

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

morph.

at $\underline{\underline{\text{most } 1}}$ lift

Our situation: $K = k(X)$

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

inclusion of generic pt

$(= \operatorname{Spec} A$ for now$)$

$\rightarrow \operatorname{Spec} \ldots$

$\therefore$ at most one lift

Suppose $\underset{\underset{\mathfrak{R}}{\parallel}}{x} \in X$ s.t. $\mathcal{O}_{X,x} \subset R$.

$$\begin{tikzcd} & \mathcal{O}_{X,x} \arrow[d, phantom, "\parallel"] & \\ K & A_p \arrow[l, hook'] \arrow[dl, dashed] & A \arrow[l, hook'] \\ R \arrow[u, hook] & & \end{tikzcd}$$

I.e. obtain

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \operatorname{Spec} \mathcal{O}_{X,x} \arrow[r] \arrow[d] & X \arrow[dl] \\ \operatorname{Spec} R \arrow[ur] \arrow[r] & \operatorname{Spec} \mathbb{Z} & \end{tikzcd}$$

$\operatorname{Spec} R \ni z$ cl. pt.

thus a lift. Properly $(*)$: $z \mapsto x$

If $x' \in X$ satisfies the same, then lifts are the same b/c sep

$\therefore x = x'$ by $(*)$.

$$\begin{tikzcd} K & \mathcal{O}_{X,x} \arrow[l, hook'] \arrow[dl] \supset P \\ R \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

val'n ring

want: P maps into the max. ideal of R.

4. Show: val^n rings are integr. closed.

$R \subset K$ — val^n ring, field of fracs

$a \in K \setminus R$, a integral

$\therefore a^{-1} \in R$

$a^n + c_{n-1} a^{n-1} + \dots + c_0 = 0$

$c_i \in R$

$$\therefore a + \underbrace{c_{n-1} + c_{n-2} a^{-1} + \dots + c_0 a^{-(n-1)}}_{\in R} = 0$$

$\therefore a \in R$ ※