1. $\varphi : B \longrightarrow A \in \mathrm{CRing}$

Suppose $\exists f_1, \ldots, f_n \in B$ s.t. $\cdot (f_1, \ldots, f_n) = B$

$\cdot A_{\varphi(f_i)}$ is a f.f. B_{f_i}-module.

Show that A is a f.f. B-module.

Let $\frac{x_1^{(i)}}{f_i^{e_1^{(i)}}}, \ldots, \frac{x_{n_i}^{(i)}}{f_i^{e_{n_i}^{(i)}}} \in A_{f_i}$ ($x_{n_i}^{(i)} \in A$)

generate as a B_{f_i}-module.

Then $x_1^{(i)}, \ldots, x_{n_i}^{(i)}$ also generate (b/c $\frac{1}{f_i^e} \in B_{f_i}$ is a unit.)

Claim $\{x_j^{(i)}\}$ generate A as a B-module.

Say there are N gens.

Obtain $B^N \xrightarrow{\alpha} A$ map of B-modules.

RTP α is surjective.

STP α_{f_i} is surjective.

i.e. $\{x_j^{(i)}\}_{i,j}$ generate A_{f_i}

which is clear. $\square$

If $f_1, \ldots, f_n$ gen B, then $\alpha : M \to N \in B\text{Mod}$ is surj.

$\iff \alpha_{f_i}$ is surj. $\forall i$

Equivalently, $M = 0 \iff M_{f_i} = 0 \ \forall i$.

Let $\mathfrak{P}$ be a max. (or any proper) ideal of B.

Then $\exists i$ s.t. $f_i \notin \mathfrak{P}$. (Otherwise $f_i \in \mathfrak{m} \ \forall i$, i.e. $\mathfrak{m} = A$)

$$M_{\mathfrak{p}} = (M_{f_i})_{\mathfrak{p}} = 0$$

$\therefore$ reduce to other stalk result.

Alternatively $M_{f_i} = 0$ $\forall m \in M$,
$f_i^N \cdot m = 0$ for $N \gg 0$.

$$\mathrm{Ann}(m) = \{a \in A \mid am = 0\}$$
$$\cup$$
$$f_i^N \ \forall N \gg 0$$

$\&$ $(f_1^N, \ldots, f_n^N) = A$ (ex.)

Q. $\varphi : B \to A \in \mathrm{CRing}$. Show that

$\operatorname{Spec}\varphi = \alpha : \operatorname{Spec} A \to \operatorname{Spec} B$ is an open immersion $\iff \exists \{f_i \in B\}_{i \in I}$ s.t. $(\varphi(f_i)) = A$ & each $B_{f_i} \to A_{f_i}$ is iso.

α is an open immersion iff α is an iso onto an open subscheme.

This is local on $\operatorname{Spec} B$: If $\operatorname{Spec} B = \bigcup_i U_i$, $U_i \subset \operatorname{Spec} B$ open, then α is open imm. $\iff$ $\alpha^{-1}(U_i) \to U_i$ is open imm. $\forall i$.
(basically triv.)

Suppose α is an open imm.
$\alpha(\operatorname{Spec} A) = U \subset \operatorname{Spec} B$ open

$$\underset{\text{Covering}}{\bigcup_{i \in I} D(f_i)} \overset{\shortparallel}{} \qquad f_i \in B.$$

Then $\alpha^{-1}(D(f_i)) \to D(f_i)$ is iso $\forall i$.
(b/c restr^n of $\operatorname{Spec} A \xrightarrow[\cong]{\alpha} U$)

$B_{f_i} \cong A_{f_i}$.

$\operatorname{Spec} A = \bigcup_i D(\alpha(f_i))$

i.e. $(\varphi(f_i)) = A$ //

Conversely, suppose given $\{f_i\}_{i \in I} \subset B$.

s.t. $(\varphi(f_i)) = A$ & $A_{f_i} = B_{f_i}$ $\forall i$.

I.e.: given $D(f_i) \subset \operatorname{Spec} B$

s.t.: $\alpha^{-1}(D(f_i)) \xrightarrow{\sim} D(f_i)$

& $\operatorname{Spec} A = \bigcup_i \alpha^{-1}(D(f_i))$.

Then let $U = \bigcup_i D(f_i)$. Then $U \subset \operatorname{Spec} B$ is open

& $\alpha : \operatorname{Spec} A \to U$ is iso.

$\therefore$ α is an open imm.

Using: $\alpha : X \to Y$ mor. of schemes,

$Y = \bigcup_i U_i$, $U_i \subset Y$ open.

Then α is iso $\Leftrightarrow$ $\alpha^{-1}(U_i) \to U_i$ iso $\forall i$.

3. $X \in \mathrm{Sch}$

$$X \longrightarrow \{\text{Closed, irred. subsets of } X\}$$
$$x \longmapsto \overline{\{x\}}.$$

Show this is injective.

Clearly well-defined.

Let $x, y \in X$, $\overline{\{x\}} = \overline{\{y\}}$.

Let $U = \operatorname{Spec} A \subset X$ be open nbd. of x.

Then $y \in U$: If not, then

$$\underset{\text{closed}}{y \in X \setminus U} \therefore \underset{\substack{\cup \\ x}}{\overline{\{y\}}} \subset X \setminus U \quad \times.$$

$\therefore$ W.M.A. $X = \operatorname{Spec} A$.

$x \leftrightarrow P \subset A$ prime ideal

$$\overline{\{x\}} = \bigcap_{\substack{\operatorname{Spec} A \supset Z \ni x \\ \text{closed}}} Z = \bigcap_{I \supset P} \{Q \supset I\}$$

Claim: $\displaystyle\bigcap_{Q \in \overline{\{x\}}} Q = P.$

b/c : $P \subset Q$ $\forall$ these Q *
but also $P = Q$ is allowed.

$\therefore$ From $\overline{\{x\}}$ we can recover x uniquely.
$\therefore$ $\overline{\{x\}} = \overline{\{y\}} \implies x = y$.

4. $X \in Sch$ $\quad A = \mathcal{O}_X(X)$

X affine $\iff$ $f_1, \dots, f_n \in A$, $(f_1, \dots, f_n) = A$
$X_{f_i} = \{x \in X \mid f_i(x) \neq 0 \in k(x)\}$
is affine.

NB: $X_{f_i} \subset X$ open, so this makes sense.

$\Rightarrow$ trivial.

$\Leftarrow$: <u>Lemma</u> $f \in A$, $\mathcal{O}_X(X_f) = A_f$.

Let $\{U_i\}_{i \in I}$ be an affine open cover of X
Let $\{U_{ij}^{(k)}\}_k$ be — " — $U_i \cap U_j$.

Then using unique gluing axiom, get exact seq.

$$0 \to \underset{\overset{\shortparallel}{A}}{\mathcal{O}_X(X)} \to \prod \mathcal{O}_X(U_i) \xrightarrow{res_1 - res_2} \prod \mathcal{O}_X(U_{ij}^{(k)})$$

$i \qquad\qquad i<k$

Localize at f:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(X)_f \arrow[r] \arrow[d, equal] & \prod_i \underbrace{\mathcal{O}_X(U_i)_f}_{\overset{\shortparallel}{U_{i,f}}} \arrow[r] \arrow[d, equal] & \prod_{i<k} \underbrace{\mathcal{O}_X(U_{ik})_f}_{U_{ik,f}} \arrow[d, equal] \\
0 \arrow[r] & \mathcal{O}_X(X_f) \arrow[r] & \prod_i \mathcal{O}_X(U_{i,f}) \arrow[r] & \prod_{i<k} \mathcal{O}_X(U_{ik,f})
\end{tikzcd}$$

this is exact b/c $\bigcup_i U_{i,f} = X_f$.

NOTE: only works if covering is finite, b/c localization does not commute with infinite products "$\prod$"

To finish, recall (prob. 3, last week):

canonical map $X \xrightarrow{\alpha_X} \operatorname{Spec} A$,

X affine $\Longleftrightarrow$ α_X is iso.

May check locally on $\operatorname{Spec} A$, so X affine $\Longleftrightarrow$

$$\alpha_i : \underbrace{\alpha^{-1}(D(f_i))}_{X_{f_i}} \longrightarrow \operatorname{Spec} A_{f_i} \quad \text{iso } \forall i$$

By Lemma, $\mathcal{O}_{X_{f_i}}(X_{f_i}) = A_{f_i}$.

$\therefore \alpha_i = \alpha_{X_{f_i}}$

$\therefore \alpha_i$ is iso $\Leftrightarrow X_{f_i}$ is affine. $\square$

Correction: Apply argument from lemma to $U_i = X_{f_i}$.

Note that $U_i \cap U_j = (X_{f_i})_{f_j}$ is itself affine.

$\therefore$ need no further refinement, no problems.