1. Ex: integral domain A

$\operatorname{Frac}(A) = K$

morphism of ringed spaces $\operatorname{Spec} K \longrightarrow \operatorname{Spec} A$

which is <u>not</u> locally ringed.

$A = \mathbb{Z}_{(p)}$ (or any dvr)

$$\operatorname{Spec} A = \{ \underset{\substack{\parallel \\ (p)}}{m}, \underset{\substack{\parallel \\ (0)}}{\eta} \}$$

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{f} & \operatorname{Spec} A \\ \parallel & & \\ \{*\} & * \longmapsto & m \end{array}$$

$$f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} K}$$

$$\begin{array}{cccc} \{m, \eta\} & A & \longrightarrow & K \\ \cup\!\!\mid & \downarrow & \circlearrowleft & \downarrow \\ \{\eta\} & K & \longrightarrow & 0 \\ \cup\!\!\mid & \downarrow & \circlearrowleft & \downarrow \\ \emptyset & 0 & \longrightarrow & 0 \end{array}$$

On stalks at m:

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec} A, m} & \longrightarrow & \mathcal{O}_{\operatorname{Spec} K, *} \\ \parallel & & \parallel \\ A & \longrightarrow & K \\ m & \longmapsto & ? \not\subseteq (0) \end{array}$$

$\therefore$ <u>not</u> a local hom

$\therefore$ f is not a locally ringed space hom.

Pf:

$A \quad P \subset Q$

$\operatorname{Spec} A_Q/P \longrightarrow \operatorname{Spec} A$

Q. X scheme

K field

Exhibit bij^n

$$\operatorname{Hom}_{Sch}(\operatorname{Spec} K, X) \cong \{(x, \alpha) \mid x \in X, \ \alpha : K(x) \hookrightarrow K\}.$$

A morphism $f : \operatorname{Spec} K \longrightarrow X$

means: $|f| : \{*\} \longrightarrow |X|$

$f(*) = x \in X$

together with $f^\# : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} K}$

s.t. ...

Let $U \subset X$. $f_* \mathcal{O}_{\operatorname{Spec} K}(U) = \begin{cases} K : & x \in U \\ 0 : & x \notin U \end{cases}$.

If $x \in U$: $\mathcal{O}_X(U) \xrightarrow{f^\#(U)} f_* \mathcal{O}_K(U) = K$

$$\mathcal{O}_{X,x} \xrightarrow[f_x^\#]{} (f_*\mathcal{O}_K)_x = K$$

Hence $f^\#$ is determined by $f_x^\#: \mathcal{O}_{X,x} \to K$.

Since $f_x^\#$ is a local hom^n we have

$$f_x^\#(m_x) \subset m_K = (0)$$

i.e.

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[r, "f_x^\#"] \arrow[d] & K \\ k(x) \arrow[ur, "\exists!\, \alpha"'] & \end{tikzcd}$$

$\therefore$ we have produced an injection

$$\mathrm{Hom}_{Sch}(\operatorname{Spec} K, X) \longrightarrow \{(x,\alpha) \mid \ldots\}$$

To show bijection, let (x,α).

Let $\operatorname{Spec} A \subset X$ be affine open nbd of x.

$\therefore x \leftrightarrow P$

$$A \to A/P \to \mathrm{Frac}(A/P) = k(x) \xrightarrow{\alpha} K$$

β

Get $\underbrace{\operatorname{Spec} K \xrightarrow{\operatorname{Spec}\beta} \operatorname{Spec} A \hookrightarrow X}_{f}$.

This is the desired mor. f $\square$

3. $X \in \mathrm{Sch}$

$A \in \mathrm{Rings}$

Show that the canonical map

$$\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathrm{Rings}}(A, \mathcal{O}_X(X))$$

is a bij^n. $\quad f \longmapsto f^{\#}(\operatorname{Spec} A)$

Given $f : X \longrightarrow \operatorname{Spec} A$ obtain

$$f^{\#} : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_X$$

global sections: $\quad A \longrightarrow \mathcal{O}_X(X)$

$\rightsquigarrow$ this is the canonical map.

This is true if X is affine.

B/c we know that $\mathrm{Rings}^{op} =: \mathrm{Aff} \xhookrightarrow[\Lambda]{\text{f.f.}} \mathrm{Sch}$

proved in lecture

$A \xrightarrow{\alpha} B \in \mathrm{Rings} \rightsquigarrow \operatorname{Spec} B \to \operatorname{Spec} A \in \mathrm{Sch}$

$$(\operatorname{Spec}\alpha)^{\#}(\operatorname{Spec} A) = \alpha$$

& every morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ arises uniquely as $\operatorname{Spec}\alpha$. //

Define presheaves F, G on X via

$$F(U) = \mathrm{Hom}_{\mathrm{Sch}}(U, \operatorname{Spec} A)$$
$$G(U) = \mathrm{Hom}_{\mathrm{Rings}}(A, \mathcal{O}_X(U))$$

$\ulcorner V \subset U$

$$\begin{array}{ccc} U & \longrightarrow & \operatorname{Spec} A \\ \uparrow & & \\ V & & \end{array} \rightsquigarrow \begin{array}{c} F(U) \\ \downarrow \\ F(V) \end{array}$$

$$\begin{array}{ccc} A & \longrightarrow & \mathcal{O}(U) \\ & & \downarrow \\ & & \mathcal{O}(V) \end{array} \rightsquigarrow \begin{array}{c} G(U) \\ \downarrow \\ G(V) \end{array} \lrcorner$$

Claim that F, G are sheaves.

For F: More generally $\mathrm{Hom}_{\mathrm{Sch}}(-, Y)$ — any fixed scheme is obviously a sheaf.

For G: If $\{U_i\}$ cover U.

Given $\alpha_i : A \longrightarrow \mathcal{O}(U_i)$ compatibly,

then $\forall a \in A$, $\alpha_i(a) \in \mathcal{O}(U_i)$ compatible

$\therefore$ glue uniquely to $\alpha(a) \in \mathcal{O}(U)$

$\therefore \exists!$ map of sets $\alpha : A \to \mathcal{O}(U_i)$

s.t. $\alpha|_{U_i} = \alpha_i$.

Have a hom^n $F \longrightarrow G$ applying construction locally.

Want: (1) $F(X) \xrightarrow{\cong} G(X)$.

Know: (2) $F(U) \xrightarrow{\cong} G(U)$ for U affine.

Since X admits affine open covers, (2) $\Rightarrow$ (1).

$\square$

Suppose given $\varphi : A \longrightarrow \mathcal{O}_X(X)$.

$X \longrightarrow \operatorname{Spec} A$ φ_A

$$\begin{aligned} &\ \ \ \Downarrow \qquad\qquad\qquad\qquad\qquad \rightarrow \mathcal{O}_{X,x} \supset m_x \\ &x \longmapsto \varphi_x^{-1}(m_x) \end{aligned}$$

$$f^\# : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow f_* \mathcal{O}_X$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] \\ A_{f(x)} \arrow[r] & \mathcal{O}_{X,x} \end{tikzcd}$$

Given $x \in X$, $A_{f(x)} \longrightarrow \mathcal{O}_{X,x}$ ... ? ...

probably works

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4. X a scheme, $A = \mathcal{O}_X(X) \in \mathrm{Rings}$.

Show: X affine $\iff$ canonical map $X \to \operatorname{Spec} A$ is an iso.

The canonical map $X \xrightarrow{c} \operatorname{Spec} A$

corresponds to $A \xrightarrow{id} A$.

Show: X affine $\iff$ c iso.

c iso $\implies$ X affine.

X affine $\implies$ By ex 3, $\exists!\, c : X \longrightarrow \operatorname{Spec} A$

s.t. $c^\#(\operatorname{Spec} A) : A \to \mathcal{O}_X(X) = A$ is the identity.

Let $d : X \longrightarrow \operatorname{Spec} A$ be the identity.

Then $d^\#(\operatorname{Spec} A) = id$.

$\therefore\ d = c$

$\therefore\ c$ is iso. $\square$