1. A comm. ring.

$P \in \mathrm{Spec}(A)$. Show: $\{P\}$ is closed $\iff$ P maximal.

Def.: $V \subset \mathrm{Spec}\, A$ is closed $\iff$ $V = V(I)$ for some $I \triangleleft A$

$= \{P \mid P \supset I\}$.

$\{P\}$ closed $\iff \exists I$ s.t. (1) $P \supset I$

(2) $Q \supset I$ (Q prime) then $Q = P$.

If P maximal: take $I = P$. Then $P \supset I$ ✓

$Q \supset I = P$ then $Q = P$ by max. ✓

$\therefore \{P\}$ closed

If P not maximal, i.e. $\exists Q \supsetneq P$.

maximal, so prime

Thus $P \supset I \Rightarrow Q \supset I$

$\therefore$ (2) cannot hold.

$\therefore \{P\}$ is not closed. $\square$

2. $* = 1$ pt top. space

(1) Describe $PSh(*)$, Show $Sh(*) \simeq Set$.

Open subsets of $*$ are $* \supset \emptyset$.

$\therefore$ $PSh(*) \simeq Fun(\Delta^1, Set)$

$F \mapsto \begin{pmatrix} F(*) \\ \downarrow \\ F(\emptyset) \end{pmatrix}$.

= Category with objects (A, B, α)

$A, B \in Set$, $\alpha : A \to B$

- mor:
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & A' \arrow[d, "\alpha'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}.$$

Sheaf condition only requires $|F(\emptyset)| = 1$.

$$Shv(*) \xrightarrow{\Gamma} Set$$
$$F \longmapsto F(*).$$

Claim: this is the desired equivalence.

Proof: $c : Set \longrightarrow Shv(*)$

$E \longmapsto c_E$, $\quad c_E(*) = E$
$\qquad\qquad\qquad\qquad \downarrow \exists!$
$\qquad\qquad\qquad c_E(\emptyset) = *$

Need natural isos.

$c\Gamma \overset{\eta}{\Longrightarrow} id_{Shv(*)} \qquad \Gamma c \overset{\mu}{\Longrightarrow} id_{Set}$

$$c_{\Gamma_F} \xrightarrow{\eta_F} F$$

$$c_{\Gamma_F}(*) = F(*) \longrightarrow F(*)$$

$$C_{\Gamma F}(\emptyset) = * \xrightarrow[\exists!]{\cong} F(\emptyset)$$

terminal

Clear: $\eta_F : C_{\Gamma F} \xrightarrow{\cong} F$

η_F is natural, i.e. given $F \xrightarrow{\alpha} G \in Shv(*)$

the following commutes:

$$\begin{tikzcd} C_{\Gamma F} \arrow[r, "\eta_F"] \arrow[d, "C_{\Gamma\alpha}"'] & F \arrow[d, "\alpha"] \\ C_{\Gamma G} \arrow[r, "\eta_G"'] & G \end{tikzcd}$$

Similarly for μ. $\square$

(2) $X \in Top$ $\quad x \in X$, $\pi : * \longrightarrow X$ image $\{x\}$.

$\pi_* : Set \overset{c}{\simeq} Shv(*) \longrightarrow Shv(X)$.

- Describe $\pi_* E$ for any set E
- Compute $(\pi_* E)_y$ for $y \in X$.

$U \subset X$ open

$$(\pi_* E)(U) = C_E(\pi^{-1}(U)) = \begin{cases} C_E(*) & \text{if } x \in U \end{cases}$$

Restriction $(\pi_x E)(U) \to (\pi_x E)(V)$

is
- identity if $x \in V \subset U$
- else the unique map.

$$\begin{cases} c_E(\emptyset) & \text{if } x \notin U \\ \,\,\Vert \\ \,\,* \end{cases}$$

NB: this is called a skyscraper sheaf

Stalks: $(\pi_x E)_y = \underset{y \in U}{\operatorname{colim}}\ \pi_x E(U)$

$$= \begin{cases} E & \text{if every open nbd. of } y \text{ contains } x \\ * & \text{else: } \underbrace{\exists \text{ open nbs. of } y \text{ not cont } x}_{*} \end{cases}$$

NB: $* \Leftrightarrow \exists\, V \subset X$ closed, $x \in V$, $y \notin V$

$\Leftrightarrow \overline{\{x\}} \not\ni y$

$$= \begin{cases} E & \text{if } y \in \overline{\{x\}} \\ * & \text{if } y \notin \overline{\{x\}}. \end{cases} \qquad \square$$

3. $f : X \to Y \in \mathrm{Top}$, $F \in \mathrm{Shv}(X)$
$x \in X$, $y = f(x)$.

- Construct a canonical map $f_*(F)_y \to F_x$

– give ex. where not iso.

$$f_*(F)_y = \operatorname{colim}_{y \in U} (f_* F)(U) = \operatorname{colim}_{y \in U} F(f^{-1}(U))$$

$$\downarrow \quad \text{b/c } f^{-1}(U) \text{ is open nbd. of } x.$$

$$F_x = \operatorname{colim}_{x \in V} F(V)$$

$X = \{x, x'\}$ (discrete top.)

$\downarrow f$

$Y = \{y\}$

$$f_* F_y = (f_* F)(\{y\}) = F(\{x, x'\}) = F(\{x\}) \times F(\{x'\})$$

$$F_x = F(\{x\})$$

our canonical map

$\rightsquigarrow$ induced by rest^n along

$\{x\} \hookrightarrow \{x, x'\}$

i.e. it is the proj^n.

$\therefore$ iso iff $|F(\{x'\})| = 1$

May choose $F(\{x'\})$ at will …

$F \in \mathrm{Shv}(X)$

$F(\{x\}) = \{1, 2\}$

$F(\{x'\}) = \{3, 4\}$

$F(\emptyset) = *$

$$F(X) = \{1,2\} \times \{3,4\}$$

- this is a sheaf
- map not iso.

4. Show that $F \longmapsto X_F$ induces equiv. of cat. between $Shv(X)$ & local homeo^n over X.

Def^n $\alpha : E \to X$ is a local homeo^n if $\forall e \in E \; \exists \, e \in U \underset{\text{open}}{\subset} E$ s.t. $\alpha(U) \subset X$ is open & $\alpha : U \longrightarrow \alpha(U)$ is a homeo^n.

Def^n $\acute{E}t(X)$ is category with objects $(E, \alpha : E \to X)$ where α is a local homeo^n, & morphisms from (E, α) to (E', α') are cts. maps $f : E \to E'$ s.t.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & E' \arrow[dl, "\alpha'"] \\ X & \end{tikzcd}$$

commutes.

Construction $F \in Shv(X)$

$$F_X = \coprod_{x \in X} F_x$$ ← stalk, a set.

Give this the finest topology s.t. $\forall U \subset X$ open, $\sigma \in F(U)$, $\hat{\sigma} : U \longrightarrow F_X$ is cts.

$$x \longmapsto \sigma_x \quad (x \in X)$$

I.e.: $V \subset F_X$ is open $\iff \hat{\sigma}^{-1}(V) \subset X$ open for all U, σ as above.

Claim: $F_X \xrightarrow{\pi} X$ is a local homeom^n.

Proof: Let $(x, \sigma_0) \in F_X$ i.e. $x \in X$, $\sigma_0 \in F_x$.

Pick U open nbd of x in X, $\sigma \in F(U)$ s.t. $\sigma_x = \sigma_0$.

Set $V = \hat{\sigma}(U) \subset F_X$.

$\pi(V) = U$ is open.

If $\sigma' \in F(U')$, then $(\hat{\sigma}')^{-1}(V) \overset{?}{=} \{ y \in X \mid \sigma'_y = \sigma_y \}$.

This is open by def^n of stalk.

$$\boxed{(\hat{\sigma}')^{-1}(V) = \{ y \in X \mid \hat{\sigma}'(y) \in V \}}.$$

$\therefore V$ is open

$\sigma : U \longrightarrow V$ $\quad \sigma$ is cts by def^n

$\pi : V \longrightarrow U$ $\quad \sigma, \pi$ are inverse bij^{ns}.

$\therefore$ remains to prove that π is cts.

I.e.: $U \subset X$ open

$U' \subset X$ open

$\sigma \in F(U')$

need: $\hat{\sigma}^{-1}(\pi^{-1}U) \subset X$ open ✓

$= U \cap U'$ $\square$

Let $\alpha : F \longrightarrow G \in Sh(X)$

Get $F_\alpha : F_X \longrightarrow G_X$. – clearly a map of sets

$F_X = \coprod_{x \in X} F_x$, $G_X = \coprod_{x \in X} G_x$

mapping $F_x \xrightarrow{\alpha_x} G_x$

Claim: F_α is cts.

Proof: Let $V \subset G_X$ open, $U \subset X$ open,

$\sigma \in F(U)$.

Need $F_\alpha^{-1}(V)$ open

i.e. $\underbrace{\hat{\sigma}^{-1} F_\alpha^{-1}}_{= \widehat{\alpha(\sigma)}^{-1}}(V)$ open

$\widehat{\alpha(\sigma)}^{-1}(V)$ open b/c V open. //

I.e.: $F \longmapsto \widehat{F}_X$

is a functor $Shv(X) \longrightarrow Et(X)$.

Remains to prove that this is an equivalence.

Construction $\pi : E \to X \in \acute{E}t(X)$, $U \subset X$

$$\Gamma(U, E) = \text{sections of } E \text{ over } U$$

$$= \text{cts maps } \sigma : U \to E \text{ s.t.}$$

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ U \arrow[ur, "\sigma"] \arrow[r, hook] & X \end{tikzcd}$$

If $V \subset U$, $\sigma \in \Gamma(U, E)$, then $\sigma|_V \in \Gamma(V, E)$.

$\therefore \Gamma(-, E)$ is a presheaf.

Claim $\Gamma(-, E)$ is a sheaf.

Pf: essentially tautological: given an open cover $\{U_i\}$ of a top. space X & $f_i : U_i \to Y \leftarrow$ some top. space agreeing on intersections, $\exists!\, f : X \to Y$.

If $\sigma_i : U_i \to E$ section then gluing $\sigma : X \to E$ is also a section. //

Construction Given

$$\begin{tikzcd} E \arrow[rr, "q"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd} \in \acute{E}t(X),$$

$\Gamma(U,\alpha) : \Gamma(U,E) \longrightarrow \Gamma(U,E')$.

$$\sigma \longmapsto \alpha \circ \sigma$$

This defines a morphism of (pre)sheaves.

Hence functor $\Gamma : \acute{E}t(X) \longrightarrow Shv(X)$.

Claim this is an inverse.

Lemma 1 $\quad \Gamma(-,E)_x = \pi^{-1}(\{x\})$

Lemma 2 If $\left(\begin{smallmatrix} E \\ \downarrow\pi \\ X \end{smallmatrix}\right) \in \acute{E}t(X)$, then the top. on E is the finest s.t. $\forall U \subset X$, $\sigma \in \Gamma(U,E)$ the map $\sigma : U \to E$ is cts.

Finish proof: $F \in Shv(X)$. Define morphism

$$\Gamma(-,F_x) \leftarrow F$$

over U: $\hat{\sigma} : U \to F_x \longleftarrow\!\shortmid\ \sigma \in F(U)$.

- this is well-defined
- this is natural in F

[i.e. for $F \to G$,
$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \circlearrowleft & \downarrow \\ \Gamma(-,F_x) & \to & \Gamma(-,G_x) \end{array}$$
]

– this is an iso: check on stalks & use that

$\Gamma(-, F_X)_x = F_x$ by Lemma 1 & construction of F_X.

$E \in \bar{E}t(X)$. Define

$$\Gamma(-,E)_X \xrightarrow{\cong} E$$

$$\Gamma(-,E)_X = \coprod_{x\in X} \Gamma(-,E)_x \overset{LA1}{=} \coprod_{x\in X} E_x$$

bij^n of sets ⇝ homeom^m by Lemma 2

– this is natural in E

$$\therefore\ Shv(X) \underset{\Gamma(-,E) \leftarrow E}{\overset{F \mapsto F_X}{\rightleftarrows}} \bar{E}t(X)$$

are indeed inverse equivalences. $\square$

Summary of the steps

$Shv(X)$ ← understand this

$Et(X) \leftarrow$ some new category,

namely objects are $\begin{matrix} E \\ \downarrow \pi \\ X \end{matrix}$ spaces over X s.t. ...

$*$ A section of π means: a cts map $\sigma : X \to E$ s.t. $\pi\sigma = id$

Clever idea: there is a functor $Shv(X) \to Et(X)$

$$F \longmapsto F_X.$$

How does this work?

As a set, $F_X = \coprod_{x \in X} F_x$.

If $\sigma \in F(U)$, obtain $\hat{\sigma} : U \longrightarrow F_X$.

$$x \longmapsto \sigma_x$$

Want a topology on F_X s.t.

sections of F $\longleftrightarrow$ sections of $\begin{matrix} F_X \\ \downarrow \\ X \end{matrix}$

i.e. $F(U)$ i.e. $*$.

Need: each $\hat{\sigma}$ is cts.

Obs.: $\exists$ a finest topology with this property.

Rest of proof: this finest topology achieves the goal

– the induced functor is actually equiv. of cat.