1. $X \in \mathrm{Top}$ $\qquad \underset{\sim}{S}, \underline{S} \in PSh(X)$

$S \in \mathrm{Set}$ $\qquad \underset{\sim}{S}(U) = S$

$\qquad \underline{S}(U) = C(U, S)$

$$\underset{\sim}{S} \xrightarrow{\alpha} \underline{S}$$

NB: this is "take the ass. sheaf" ∴ iso on stalks

$$\begin{array}{ccc} \alpha(U) : \underset{\sim}{S}(U) & \longrightarrow & \underline{S}(U) \\ \| & & \| \\ S & & C(U,S) \\ \rotatebox{90}{$\in$} & & \\ s & \longmapsto & c_s \end{array} \quad , \quad c_s(u) = s \ \forall u \in U.$$

Let $x \in X$. Show: $\alpha_x : \underset{\sim}{S}_x \to \underline{S}_x$ is an iso.

$$\begin{array}{ccc} \underset{\sim}{S}_x & \longrightarrow & \underline{S}_x \\ \wr\| & \swarrow_{\beta} & \\ S & & \end{array}$$

Define $\beta(f) = f(x)$. This is well-defd ✓

$$\beta \alpha_x(s) = \beta(c_s) = c_s(x) = s$$

$\therefore \beta \alpha_x = id$.

Let $U \subset X$ open nbd. of x, $f : U \to S$ cts.

$t = f(x)$. $\quad \{t\} \subset S$ open

$\therefore U' = f^{-1}(\{t\})$ is also open nbd of x

$f|_{U'} = c_t$

$\therefore [f] = \alpha_x \beta [f] \in \underline{S}_x$

$\therefore\ \alpha\circ\beta = id$

Hence: α iso (w/ inverse β). $\square$

Show that α is not an iso unless $|S|=1$.

$\emptyset\subset X$ $\quad \underset{\sim}{S}(\emptyset) = S$

$\underline{S}(\emptyset) = C(\emptyset, U) = \{*\}$

α iso $\Rightarrow$ $|S| = 1$

$|S| = 1 \Rightarrow C(-,S) = \{*\}$
$= S$

//

2. X Aff. variety / alg. closed field k.

$\mathcal{O} : U \mapsto \mathcal{O}(U)$
sheaf of regular functions.

$x\in X \longleftrightarrow m_x \subseteq A = \mathcal{O}(X)$.

Show: $\mathcal{O}_x \cong A_{m_x}$.
stalk $\qquad$ loc^n

Pf $\underline{\text{Lemma}}$ If $X\in\text{Top}$, $x\in X$, $\mathcal{U}$ a basis of $\overset{\text{op.}}{\text{nbds}}$ of x
(i.e. if $U\subset X$ $\overset{\text{op}}{\text{nbd.}}$ nbd of x then $\exists U'\in\mathcal{U}$
with $U'\subset U$).

Then for any presheaf F,

$$F_x = \operatorname{colim}_{U \in \mathcal{O}} F(U) = \coprod_{U \in \mathcal{O}} F(U) \Big/ \sim .$$

omitted $\to$ $\square$

$U \subset X$ open nbd. of x

$$U = X \setminus Z(f_1, \dots, f_n) = \bigcup_i D(f_i)$$

$f_i(x) \neq 0$.

$\therefore$ $\mathcal{U} = \{ D(f) \mid f(x) \neq 0 \}$ form basis of open nbd. of x.

Known: $\mathcal{O}(D(f)) = A[f^{-1}]$.

$$\therefore \mathcal{O}_x = \operatorname{colim}_{f(x) \neq 0} A[f^{-1}]$$

Let $K = \operatorname{Frac}(A)$

$$\downarrow \cong$$

$$\bigcup_{f(x) \neq 0} A[f^{-1}] \subset K$$

In contrast, $A_{m_x} \subset K$

$$A_{m_x} = \left\{ \frac{a}{f} \mid f \notin m_x \right\} = \left\{ \frac{a}{f} \mid f(x) \neq 0 \right\}$$

(Recall: $f(x) = 0 \iff f \in m_x$)

$\therefore$ desired iso.

3. $F \in \cancel{\mathrm{Shv}}(X)$ PSh , $s \in F(X)$.

$$\mathrm{Sup}(s) = \{x \in X \mid s_x \neq 0 F_x\} \subset X.$$

Show: $\mathrm{Sup}(s)$ is closed in X.

Equiv, $X \setminus \mathrm{Sup}(s)$ is open

$$= \{x \in X \mid s_x = 0 \in F_x\}$$

$s_x = g_x \in F_x \iff \exists U \ni x,\ s|_U = g|_U$ (def^n of Shlh)

$\therefore s_x = 0 \Rightarrow \exists U_x \ni x$ s.t. $s|_U = 0$

$\Rightarrow s_y = 0 \ \forall y \in U_x$

$\bigcup_{x \mid s_x = 0} U_x$ is a union of open sets, so open. $\square$

4. $F, G \in PSh(X)$, $x \in X$

(1) Show: $(F \times G)_x \cong F_x \times G_x$

$$(F \times G)_x = \coprod_{U \ni x} F(U) \times G(U) \Big/ \sim \quad \text{i.e. } \{(s_U, t_U)\}^{x \in U \subset X}_{\substack{s_U \in F(U) \\ t_U \in G(U)}}$$

$$F_x \times G_x = \Big(\coprod_{U \ni x} F(U) \big/ \sim \Big) \times \Big(\coprod_{U \ni x} G(U) \big/ \sim \Big)$$

i.e. (s_U, t_V)

$x \in U \subset X \quad s_U \in F(U)$

$x \in V \subset X \quad t_V \in F(V)$.

Map $\varphi : (F \times G)_x \longrightarrow F_x \times G_x$ ~ well-defined ✓

$(s_U, t_U, U) \longmapsto (s_U, t_U, U, U)$

$\psi : F_x \times G_x \longrightarrow (F \times G)_x$

$(s_U, t_U, U, V) \longmapsto (s_U|_{U \cap V}, t_U|_{U \cap V}, U \cap V)$

well-defined? ✓ If U', V' $\quad U'' \subset U' \cap U$

$s_{U'}, s_{V'} \quad V'' \subset V' \cap V$

$s_{U'}|_{U''} = s_U|_{U''}$ & $t_V|_{V''} = t_{V'}|_{V''}$

$$\psi((s_U, t_V, U, V)) \overset{?}{=} \psi((s_{U'}, t_{V'}, U', V'))$$

$$\begin{array}{ccc} \| & & \| \\ (s_U|_{U \cap V}, t_V|_{U \cap V}, U \cap V) & & (s_{U'}|_{U' \cap V'}, t_{V'}|_{U' \cap V'}) \\ \| & & \| \end{array}$$

restrict to $U'' \cap V'' \subset U \cap V$ $\quad \cap U' \cap V'$

$$(s_U|_{U'' \cap V''}, t_V|_{U'' \cap V''}, U'' \cap V'') = \dots$$

Check that $\psi \varphi = id$ ✓

$$\varphi\psi^{-1}((s_U, t_V, U, V)) = (s_U|_{U\cap V}, t_V|_{U\cap V}, U\cap V, U\cap V)$$
$$= (s_U, t_U, U, V)$$

$\therefore \varphi, \psi$ inverse iso. $\square$

(2) If F, G are sheaves on X, then G_0 is $F \times G$.

Let $\{U_i\}$ an open cover of U,

$(s_i, t_i) \in (F\times G)(U_i)$

s.t. $(s_i, t_i)|_{U_i \cap U_j} = (s_j, t_j)|_{U_i\cap U_j}$.

RTP: $\exists! (s,t) \in (F\times G)(U)$ s.t. $(s,t)|_{U_i} = (s_i, t_i)$.

$\{s_i\}$ are compat. sect. of F
$\{t_i\}$ — " — G

$\therefore \exists!$ $s \in F(U)$, $t \in G(U)$ s.t. $\left.\begin{array}{l} s|_{U_i} = s_i \\ t|_{U_i} = g_i \end{array}\right\}$ // same cond^n $\square$

Category theory:

$$a : PSh(X) \underset{\text{"right adjoint"}}{\overset{\text{incl.}}{\rightleftarrows}} Shv(X) : i$$

ass. sheaf "left adjoint"

Th^m: if $L \dashv R$ then L pres. all colim
R pres. all limits.

Th^m In category of Sets, filtered colimits commute with finite limits.

Partially ordered set I (e.g. I = open nbd. of $x \in X$ ordered by rev. incl^n)
is called filtered if
given $x, y \in I$ $\exists$ $z \in I$ $z \geq x$, $z \geq y$ (e.g. U, V nbd. $\Rightarrow U \cap V$ nbd.)

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what about $\mathcal{O}(D(f))$?

$X \subset \mathbb{A}^n$
$\shortparallel$
$Z(p_1, \dots, p_r)$

$$D(f) = X \setminus Z(f) \cong X_f \subset \mathbb{A}^{n+1}_{x_1, \dots, x_n, T}$$

as variety

$$X_f = Z(p_1, \dots, p_r, f \cdot T - 1)$$

$$\therefore \mathcal{O}_X(D(f)) \cong \mathcal{O}_{X_f}(X_f) = \mathcal{O}_X(X)[T]/_{fT-1}.$$

X variety, $\mathcal{O}_X$ = structure sheaf
= sheaf of regular functions

Claim: If $X \cong Y$ as varieties
then $X \cong Y$ as top. spaces
$\mathcal{O}_X \in Shv(X) \cong Shv(Y) \ni \mathcal{O}_Y$

Claim: $f \in A$ then $A[f^{-1}] \cong A[T]/_{Tf-1}$.