1. A an integral domain, field of fractions K.

Recall that $A \subset K$

& $S^{-1}A \subset K \quad \forall\, S$ mult. subset.

Show: $\bigcap_{\substack{m \\ \text{max. ideal} \\ \text{of } A}} A_m = A$ as subset of K.

Clear: $A \subset A_m \quad \forall m$

$\therefore A \subset \bigcap_m A_m$.

Conversely, let $f \in K$.

Put $I = \{ a \in A \mid a \cdot f \in A \}$.

This is an ideal of A

Moreover, $I = A \iff f \in A$.

Suppose $f \in A_m$. Then $I \not\subset m$.

Indeed $f = \frac{a}{b}$, $b \in A \setminus m$, & so $b \in I \setminus m$.

If $f \in \bigcap_m A_m$, then $I \not\subset m \quad \forall m$

$\therefore I = A$

$\therefore f \in A$. $\square$

2. $\varphi : \mathbb{A}^1_k \longrightarrow Z(X^3 - Y^2) \subset \mathbb{A}^2_k$

$T \longmapsto (T^2, T^3)$.

Show: this is a mor. of varieties.

Q: Is this an iso?

Construction: Suppose X is a quasi-projective variety, $f_1, \dots, f_n$ are regular functions on X.

Put $F = (f_1, \dots, f_n) : X \longrightarrow \mathbb{A}^n_k$.

Let $Z \subset \mathbb{A}^n_k$ be locally closed s.t. $F(X) \subset Z$.

Then $F : X \longrightarrow Z$ is a morphism of varieties.

Proof: f_i is cts. $\therefore$ F is cts.

Let $U \subset Z$ open, $g : U \longrightarrow \mathbb{A}^1$ be regular.

RTP: $gF : F^{-1}(U) \longrightarrow \mathbb{A}$ is regular.

WMA: $g = \frac{p}{q}$, where $p, q : \mathbb{A}^n \longrightarrow \mathbb{A}^1$ regular

& $q(z) \neq 0$. Similarly

Then $gF = \frac{p(f_1, \dots, f_n)}{q(f_1, \dots, f_n)} \in \mathcal{O}(F^{-1}U)$ is regular. //

Since —〃— is a k-algebra

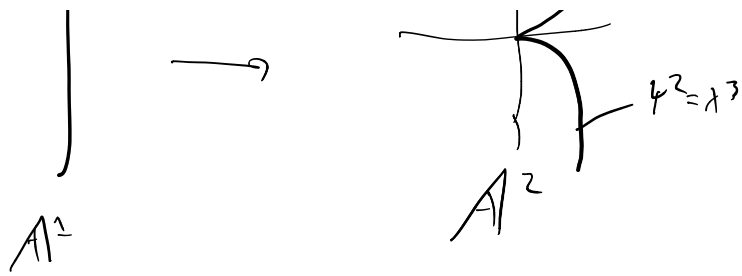
$T^2, T^3 \in \mathcal{O}(\mathbb{A}^1) = k[\mathbb{A}^1]$, $\quad (T^2)^3 - (T^3)^2 = 0$

$\therefore \varphi(\mathbb{A}^1) \subset Z(X^3 - Y^2) = Z$

& $\varphi : \mathbb{A}^1 \longrightarrow$ —〃— —

is a morphism by the construction.

In pictures:



Have $\varphi^*: k[Z] \longrightarrow k[\mathbb{A}^1]$

$$\begin{array}{ccc} k[Z] & \longrightarrow & k[\mathbb{A}^1] \\ \| & & \| \\ \mathcal{O}(Z) & & \\ \| & & \\ \dfrac{k[X,Y]}{\sqrt{Y^2-X^3}} & \longrightarrow & k[T] \end{array}$$

$X \longmapsto T^2$
$Y \longmapsto T^3$

This is <u>not</u> surjective, $\therefore \varphi^*$ is not iso
$\therefore \varphi$ is not iso. $\square$

3. Show that

$$\varphi : Z = Z(XY-1) \longrightarrow \mathbb{A}^2_k \setminus 0$$
$$\cap$$
$$\mathbb{A}^2$$
$$(X,Y) \longmapsto X$$

is a morphism of varieties.
Is it an iso?

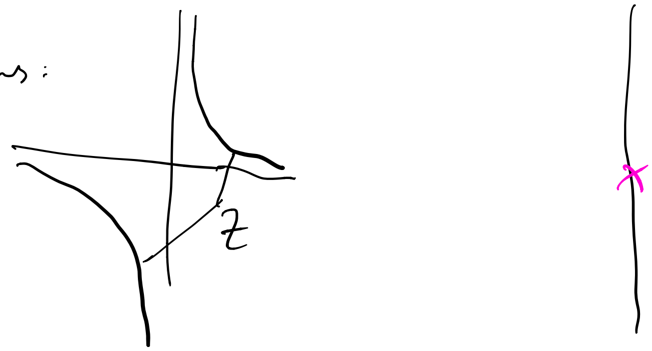
$$Z \hookrightarrow \mathbb{A}^2 \xrightarrow{(X,Y) \mapsto X} \mathbb{A}^1$$ is a regular function

[b/c: regular function = morphism to $\mathbb{A}^1$

& $Z \hookrightarrow \mathbb{A}^2$ is a morphism,
$\mathbb{A}^2 \to \mathbb{A}^2$ is regular $]$

$\varphi(Z) \subset \mathbb{A}^1 \setminus 0$ $\therefore$ φ is a morphism by the construction in prob. 3.

In pictures:



$$\psi : \mathbb{A}^1 \setminus 0 \to Z \qquad x \longmapsto \left(x, \tfrac{1}{x}\right)$$ is an inverse morphism.

$\varphi\psi x = x$

$\psi\varphi(x,y) = \left(x, \tfrac{1}{x}\right) = (x,y)$

use construction

need: $\psi_1, \psi_2 : \mathbb{A}^1 \setminus 0 \to \mathbb{A}^1$

$x \longmapsto x$

$x \longmapsto 1/x$

are regular ✓

4. Let X be a quasi-projective variety. Show that X admits a cover by open affine subvarieties.

$X \subset \mathbb{P}^n$. $\quad \mathbb{P}^n = \bigcup \mathbb{A}^n$

$$\therefore X = \bigcup \underbrace{X \cap \mathbb{A}^n}_{\text{quasi-affine}}$$

$\therefore$ WMA X quasi-affine

Say $X = Z(f_1, \dots, f_n) \setminus \underbrace{Z(g_1, \dots, g_m)}_{= Z(g_1) \cap \dots \cap Z(g_m)} \subset \mathbb{A}^r$.

$$= \bigcup_i \underbrace{\left[Z(f_1 \dots f_n) \setminus Z(g_i) \right]}$$

STP: V is affine.

Consider $W \subset \mathbb{A}^{r+1}$ coord. $x_1, \dots, x_r, T$

$$= Z(f_1, \dots, f_n, g_i \cdot T - 1)$$

Shall show $V \cong W$.

$$\varphi : W \longrightarrow V$$
$$(x_1, \dots, x_r, T) \longmapsto (x_1, \dots, x_n)$$
is a mor. by the cons^n.

$$\psi : V \longrightarrow W$$
$$(x_1 \dots x_r) \longmapsto \left(x_1, \dots, x_r, \frac{1}{g_i(x_1, \dots, x_r)}\right)$$
is a mor. by const^n.

$\varphi \psi = id$
$\psi \varphi = id$ as before.

$\therefore \varphi, \psi$ are inverse morphisms

$\therefore V$ is affine as needed. $\square$