1. (1) $I \subset S = k[X_0, \dots, X_n]$ homogeneous.

Show: $\sqrt{I}$ is also homog.

$f = f_0 + f_1 + \dots + f_d \in S$ decomp^n into homog. cpts

$f \in \sqrt{I}$.

i.e. $f^n \in I$ for some n

homog. cpt. of degree $n \cdot d$ $= f_d^n$

I homog. $\Rightarrow f_d^n \in I$

$\Rightarrow f_d \in \sqrt{I}$.

$\therefore f - f_d = f_0 + \dots + f_{d-1} \in \sqrt{I}$

Now induct on deg f.

$f \in \sqrt{I}$ (alternative sol^n)

$\lambda \in k^\times$

$f_\lambda(x) := f(\lambda x) \overset{?}{\in} \sqrt{I}$

$f^n \in I$

$f_\lambda^n = (f^n)_\lambda \in I$

b/c I homg.

(2) $I \triangleleft S$ homg., $\sqrt{I} = S_+$.

Show: $I_n = S_n$ for $n \gg 0$.

$S_+ = (X_0, \dots, X_n)$

$= \sqrt{I}$

$\therefore \exists d$ s.t. $X_i^d \in I \quad \forall i$

$\therefore \exists e$ s.t. $(X_0, \dots, X_n)^e \subset I$ (take $e = d^{n+1}$)

$S_+^e = S_{\geq e}$

2. Determine maximal proper homogeneous ideals in S.

NB: not homog. max. ideals

$A = \{\text{homog. proper radical ideals}\} \setminus S_+$ — max elt. P

$\updownarrow$ — $\updownarrow$

$\{\text{closed subsets of } \mathbb{P}^n\}$ — $\{(x_0 : \dots : x_n)\}$ min elt. $= Z$

$\updownarrow$ — $\updownarrow$

$\{\text{closed } \mathbb{G}_m\text{-inv. subsets of } \mathbb{A}^{n+1} \setminus 0\}$ — min. elt. $\pi^{-1}(Z)$

max. elts of A are $\left\{ \underbrace{I^h(\{(x_0 : \dots : x_n)\})}_{(x_j X_i - x_i X_j)_{i,j}} \;\middle|\; (x_0 : \dots : x_n) \in \mathbb{P}^n(k) \right\}$

claim: $\{\text{ideals we are looking for}\}$
$= \{\text{max. elts. of } A\} \cup S_+$

Pf If I is one of our ideals,

consider $\underset{\text{homg. proper}}{I \subset \sqrt{I}}$. $\therefore I = \sqrt{I}$ by maximality.

$\square$

3. $\overline{\{XY=1\}}$ A , $\overline{\{X^2+Y^2=1\}}$ B $\subset \mathbb{P}^2(k)$
$\overline{\{Y-X^2=0\}}$ C

are related by a projective linear transformation

$\underline{A \leftrightarrow B}$: $(X+iY)(X-iY) = X^2+Y^2$

$\therefore$ matrix $\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & i & -i \end{pmatrix}$ will do the job

$\underline{\underline{\text{if}}}$ char. $\neq 2$. [NB: if char $=2$: this is not true]

Detail: M defines an a

$$\begin{array}{ccc} a: \mathbb{P}^2 & \xrightarrow{\sim} & \mathbb{P}^2 \\ \cup & & \cup \\ \mathbb{A}^2 & \xrightarrow{\cdot M} & \mathbb{A}^2 \end{array} \qquad a(\mathbb{A}^2) = \mathbb{A}^2$$

$$a(\bar{A}) = \overline{a(A)} = \overline{MA} = \bar{B}$$

PuS 4: $\bar{A} = Z^4(XY-Z^2)$ $\qquad \bar{B} = Z^4(X^2+Y^2-Z^2)$

$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$

$$\bar{A} \cap \mathbb{P}^1 = Z^4(XY-Z^2, Z)$$
$$= \{(x:y:0) \mid xy=0\}$$
$$= \{(1:0:0), (0:1:0)\}$$

$$\bar{B} \cap \mathbb{P}^1 = Z^4(X^2+Y^2-Z^2, Z)$$
$$= \{(x:y:0) \mid x^2+y^2=0\}$$
$$= \{(1:\pm i:0)\}$$

Explicit $(x, \frac{1}{x}) \longleftrightarrow (x : \frac{1}{x} : 1) = (x^2 : 1 : x) \in A$

A $\quad\quad\quad$ $\therefore$ "$x \to 0$" $(0:1:0) \in \bar{A}$

$A \leftrightarrow C$ $\quad$ ex 4: $\bar{A} = Z^p(XY - Z^2)$

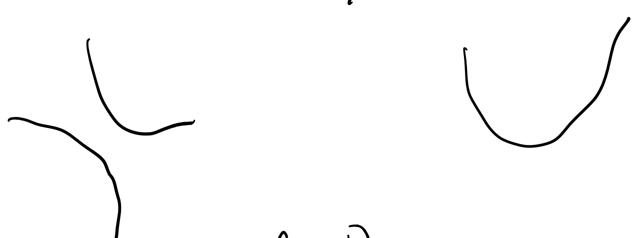
$$\bar{C} = Z^p(YZ - X^2)$$

Same as 4, just $Z \longleftrightarrow X$

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$ will do what we want. ✓

Comment: $N(\mathbb{A}^2) \not\subset \mathbb{A}^2$

in $\mathbb{A}^2$:



$A \quad \mathbb{A}^2 \setminus 0 \quad k[x, x^{-1}] \qquad C \quad \mathbb{A}^1 \quad k[x]$

" $Z(XY - 1) \;\&\; Z(Y - X^2)$ are not isomorphic varieties (so not related by a linear transf.) "

4. $f \in k[x_1, \ldots, x_n]$ degree d,

$f = f_0 + f_1 + \ldots + f_d \qquad f_i =$ homog. cpt. of degree i

degree.

$f_d \neq 0$.

$$f^h(X_0, \dots, X_n) = \sum_{0 \le i \le d} X_0^{d-i} f_i(X_1, \dots, X_n)$$

Show: $\overline{Z(f)} = Z^h(f^h)$.

Obsv: $Z(f) \subset Z^h(f^h)$ (b/c $f^h(1, X_1, \dots, X_n) = f$)

$\therefore \overline{Z(f)} \subset Z^h(f^h)$.

Note: $(fg)^h = f^h \cdot g^h$

$\overline{Z(fg)} = \overline{Z(f) \cup Z(g)} = \overline{Z(f)} \cup \overline{Z(g)}$

$Z^h(f^h g^h) = Z^h(f^h) \cup Z^h(g^h)$.

$\therefore$ WMA f irreducible.

$\therefore$ $Z(f)$ irred. dim $n-1$

$\therefore$ $\overline{Z(f)}$ irred. dim $n-1$

$\therefore \overline{Z(f)} \subset \mathbb{P}^n(k)$ is maximal proper closed irred.

Since $Z^h(f^h)$ is closed & proper,

STP $Z^h(f^h)$ is irreducible.

Enough: f^{h} is irreducible.

Suppose not: $f^{h} = p \cdot q$

$$f(x_1, \ldots, x_n) = f^{h}(1, x_1, \ldots, x_n) = p(1, x_1, \ldots, x_n) \cdot q(1, x_1, \ldots, x_n)$$

WMA everything monic & $p(1, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$

[& $q(1, x_1, \ldots, x_n) = 1$]

[This implies $q = x_0^{r}$ for some r.]

$\deg p \geq \deg f = \deg f^{h}$

$\therefore \deg q \leq 0$

$\therefore q = 1$

$\therefore f^{h}$ is irreducible as claimed