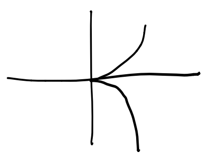
1. $\mathbb{A}^1(k) \xrightarrow{\varphi} \mathbb{A}^2(k)$

$T \longmapsto (T^2, T^3)$

Show: this is a homeom$^{\sim}$ onto

$Z = Z(X^3 - Y^2)$.

Picture:



Sol$^n$: $x^3 = y^2$. Write $x = t^2$

$\therefore y^2 = t^6$

$y = \pm t^3$

If $y = -t^3$, replace $t$ by $-t$

(does not change $x$)

$\rightsquigarrow$ all sol$^n$ of the form $(t^2, t^3)$

Conversely, $(t^2, t^3)$ always a sol$^n$.

Have shown: $\varphi(\mathbb{A}^1) = Z$

Clear: $\varphi$ cts.

Also: $\varphi$ closed. Indeed $\varphi(pt) = pt$

& all closed subsets are finite unions

of pts or all of $\mathbb{A}^1$.

Remains: $\varphi$ injective,

Proof: $\varphi(t) = 0 \iff t^2 = 0 \iff t = 0$

$\varphi(t) \neq 0 \Rightarrow t = \frac{t^3}{t^2} = \frac{y}{x}$

i.e. $\psi : Z \setminus 0 \longrightarrow \mathbb{A}^1 \setminus 0$
$(x, y) \longmapsto \frac{y}{x}$
is inverse.

$\therefore \varphi$ is cts, closed, bij$^n$ onto $Z$

$\therefore \varphi$ is homeom$^n$ — " — .

**2.** $Z = Z(XY - 1) \subset \mathbb{A}^2(k)$.

Show: $Z$ is irred. & proj$^n$ $\mathbb{A}^2(k) \xrightarrow{p} \mathbb{A}^1(k)$
$(x, y) \longmapsto x$

induces a homeom$^n$ $Z \cong \mathbb{A}^1(k) \setminus 0$.

$\mathbb{A}^1(k) \setminus 0$ is irred.

$\therefore Z \cong \mathbb{A}^1(k) \setminus 0 \Rightarrow Z$ irred.

· $p(Z) \subset \mathbb{A}^1(k) \setminus 0$:

$xy = 1 \Rightarrow x \neq 0$

Show $\varphi : Z \to \mathbb{A}^1 \setminus 0$ is cts, closed, bij$^n$.

closed: $Z \subset \mathbb{A}^2$ proper closed subset (clear)

$\therefore$ dimension at most 1

$\therefore$ proper closed subset of $Z$ = finitely ~~many~~ points

$\therefore \varphi$ closed

bij$^n$: $\psi : \mathbb{A}^1(k) \setminus 0 \longrightarrow Z$.

$$t \longmapsto (t, t^{-1})$$

is inverse to $\varphi$.

NB: $\mathbb{A}^1(k) \setminus 0 \subset \mathbb{A}^1$ is <u>open</u>

$Z \subset \mathbb{A}^2$ is <u>closed</u>

3. $\mathbb{A}^1(k) \xrightarrow{\varphi} \mathbb{A}^3(k)$

$T \longmapsto (T^3, T^4, T^5)$

Show: $\varphi$ is a homeom$^n$ onto $Z(P)$, for some prime ideal $P$ of $k[\mathbb{A}^3] = k[X, Y, Z]$ of height 2.

Suppose $\varphi$ is homeom$^m$ onto closed subset $Z$

Then $Z = Z(I)$ for $I = J(Z)$.

But $Z \cong \mathbb{A}^1(k)$ is irred, so $I = P$ prime.

Since $\dim Z = \dim \mathbb{A}^1 = 1$ & $\dim Z + ht(P) = \underbrace{\dim \mathbb{A}^3}_{=3}$

<u>$\varphi$ is homeom$^m$ onto its image $Z$</u>

- $\varphi$ is cts
- $\varphi$ is closed

Remains to prove $\varphi$ is injective.

$\varphi(t) = 0 \iff t = 0$

$\varphi(t) = (x, y, z) \neq 0$

then in fact $x \neq 0$ & $t = \frac{y}{x}$

$A^2 \setminus 0 \longrightarrow Z \setminus 0$ is isom with inverse
$(x,y,z) \mapsto \frac{y}{x}$.

$\therefore \varphi$ injective.

$Z = \varphi(A^2) \subset A^3$ is closed

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$Z = \{(T^3, T^4, T^5) \mid T \in k\}$

satisfies eqns $y^3 = x^4$.
$z^3 = x^5$

Can write $x = t^3$. Then $y^3 = t^{12}$

$\therefore y = t^4 \cdot \zeta_3^i$ primitive 3rd root of unity

$z = t^5 \cdot \zeta_3^j$

Replace $t \mapsto \zeta_3^i t$. Does not change $x$.

$y = (\zeta_3^{-i} t)^4 \zeta_3^i = \zeta_3^{-i} t^4 \zeta_3^{-i} = t^4$

$\therefore$ WMA $i = 0$

Also $y^2 = xz$.

$t^8 = t^3 \cdot \zeta_3^j t^5$

$\Rightarrow t = 0$ or $j \equiv 0 \mod 3$.

$\therefore$ WMA $j = 0$

$\therefore$ all solutions of the form $(t^3, t^4, t^5)$.

In other words, $Z = Z(y^3 - x^4, z^3 - x^5, y^2 - xz)$
is closed.

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Avoiding the calculation:

$$\mathbb{P}^1(k) \xrightarrow{\varphi} \mathbb{P}^3(k)$$

$$(T_0 : T_1) \longmapsto (T_0^5 : T_0^2 T_1^3 : T_0 T_1^4 : T_1^5)$$

$$\begin{tikzcd}
\mathbb{A}^1(k) \arrow[r, "\varphi"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{A}^3(k) \arrow[d, phantom, "\cap"] \\
\mathbb{P}^1(k) \arrow[r, "\bar{\varphi}"] & \mathbb{P}^3(k)
\end{tikzcd}$$

$$\bar{\varphi}(\infty) = \bar{\varphi}(0:1) = (0:0:0:1) \notin \mathbb{A}^3$$

$$\varphi(\mathbb{A}^1) = \bar{\varphi}(\mathbb{P}^1) \cap \mathbb{A}^3$$

$\therefore$ STP $\bar{\varphi}(\mathbb{P}^1)$ is closed.

But $\mathbb{P}^1$ is <u>proper</u>, whence its image under <u>any</u> map is closed.

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4. $\pi : \mathbb{A}^{n+1}(k) \setminus 0 \longrightarrow \mathbb{P}^n(k)$ can. map.

$Z \subset \mathbb{P}^n(k)$ closed.

$\pi^{-1}(Z) = \bigcup_i Y_i$,

the decomp$^n$ into irreducible cpts.

Show: $Y_i$ is $k^*$-invariant $\forall i$.

[I.e.: if $Z = \bigcup_i Z_i$ is decomp$^n$ into irred.,
then $\pi^{-1}(Z_i)$ is still irred.]

Proof $k[Y_i]$ is an integral domain b/c $Y_i$ irred.

$\therefore$ $k[Y_i][T]$ is also an integral domain.

$[\hookrightarrow (\mathrm{Frac}\, k[Y_i])[T]$, which is I.D.$]$

$$k[x_0, \dots, x_n] \twoheadrightarrow k[Y_i]$$

$$k[x_0, \dots, x_n, T] \twoheadrightarrow k[Y_i][T]$$

$\therefore$ $k[Y_i][T] = k[\tilde{Y}_i]$, for some $\tilde{Y}_i \subset \mathbb{A}^{n+2}(k)$

which is closed & irred.

In fact $\tilde{Y}_i = Y_i \times \mathbb{A}^1(k)$.

So: $Y_i \times \mathbb{A}^1(k)$ is irreducible.

$\therefore$ $Y_i \times \underbrace{\mathbb{A}^1(k) \setminus 0}_{k^*}$ is irreducible

$\therefore$ $Y_i \times \mathbb{A}^1_{(k)} \setminus 0 \longrightarrow \mathbb{A}^{n+1}(k) \setminus 0$

$(\underline{x}, t) \longmapsto \underline{x} \cdot t$

has irreducible image $Y_i' \subset \pi^{-1}(Z)$.

$\therefore$ $Y_i \subset Y_i' \subset Z$

inclusion of an irreducible cpt

$\therefore$ $Y_i' = Y_i$.

But $Y_i' = k^* \cdot Y_i$. $\square$

Alternative: finitely many $i$.

$x \in Y$

$t \in k^\times \Rightarrow t x \in Y$

$F: \mathbb{A}^1 \setminus 0 \longrightarrow Y$

$t \longmapsto t \cdot x$

homeom$^{m}$ onto image, which is irred.

& closed S/C ...

By def$^n$, $\exists i$ s.t. $F(\mathbb{A}^1 \setminus 0) \subset Y_i$.

Q: Suppose $Y = Y_1 \cup Y_2$ $\quad x \in Y_1 \cap Y_2$

$k^\times \cdot x \in Y_1$ but not in $Y_2$.

... maybe this does not work...?

Other alternative:

$t \in k^\times$: $t: Y \to Y$ is an autom$^m$

$\therefore k^\times$ permutes the finitely many $Y_i$.

$\therefore$ $S = \{t \in k^\times \mid tY_i \subset Y_i\}$ is infinite, since $k^\times$ is.

Now let $y \in Y_i$. Then $Sy \subset Y_i$

$\therefore \overline{Sy} \subset Y_i$ $\qquad \therefore Y_i$ is $k^\times$-inv.

$\overline{Sy} = k^\times y$

$\square$