1. $I, J \lhd R$ Show:

$$IJ \overset{\checkmark}{\subset} I \cap J \overset{\checkmark}{\subset} \sqrt{I} \cap \sqrt{J} \overset{\subset}{=} \sqrt{I \cap J} \overset{\subset}{=} \sqrt{IJ}$$

$I \subset \sqrt{I}$

$a^n \in IJ$ i.e. $a^n = \sum i \cdot j$ $i \in I$, $j \in J$

$\Rightarrow a^n \in I$ $a^n \in J$

$\therefore a^n \in I \cap J \in \sqrt{I \cap J}$

i.e. $a \in \sqrt{IJ}$

$a \in \sqrt{I} \cap \sqrt{J}$ i.e. $a^n \in I$, $a^n \in J$

$\Rightarrow a^n \in I \cap J$

$\Rightarrow a \in \sqrt{I \cap J}$

$a \in \sqrt{I \cap J}$

i.e. $a^n \in I \cap J$

i.e. $a^n \in I$, $a^n \in J$ $a^{2n} \in IJ$

$a \in \sqrt{IJ}$

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2. k infinite field

$f \in k[X_1, \dots, X_n]$ <u>not</u> 0.

Show: $\exists (x_1, \dots, x_n) = x \in k^n$ s.t. $f(x) \neq 0$.

$$k[X_1, \dots, X_n] = k[X_1, \dots, X_{n-1}][X_n]$$

:

$\therefore\ f = \sum_i f_i(x_1, \ldots, x_{n-1})\, x_n^i$

If $f_i = 0\ \forall i > 0$ then $f = f_0(x_1, \ldots, x_{n-1})$
& result follows by induction.
(start at $n=0$)

Otherwise: $f_i \neq 0$ for some i.
By induction $\exists\ (x_1, \ldots, x_{n-1}) = x'$ s.t.
$f_i(x') \neq 0$.

Consider $p(X) = f(x_1, \ldots, x_{n-1}, X)$
$= \sum_j f_j(x') \cdot X^j$ ← coeff. of $X^i \neq 0$
$\therefore\ p \neq 0$.

$\therefore$ reduce to $n = 1$.

This case holds b/c $p \in k[X]$ deg. $n \Rightarrow$
at most n roots, but k is infinite.
$\square$

Ex: $p(X) = X^p - X \in \mathbb{F}_p[X]$
then $p(x) = 0\ \forall x \in \mathbb{F}_p$

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3. X top. space.
(1) X irred., $\emptyset \neq U \subset X$ open $\Rightarrow U$ irred. & $\bar{U} = X$.
(2) $Y \subset X$. Y irred. $\Leftrightarrow \bar{Y} \subset X$ is irred.

<u>Lemma</u> top. space X is irreducible $\Leftrightarrow$ all non-empty open

subsets are closed.

Pf $\forall\, U \underset{\text{open}}{\subset} X$ dense $\iff \forall\, Z \subset X$ closed, $Z \supset U$

$\implies Z = X$

$\iff \forall\, V \subset X$ open, $V \cap U = \emptyset$

$\implies V = \emptyset$

$\iff \forall\, U, V \subset X$ open, $V \cap U = \emptyset$

$\implies V = \emptyset$ or $U = \emptyset$

$\iff X$ irred. $\square$

(1) $\emptyset \neq U \subset X$. $\quad \bar{U} = X$ by Lemma

$\emptyset \neq V \subset U \implies (\text{closure of } V \text{ in } X) = X$ by Lemma

$\text{closure of } V \text{ in } X = \bigcap_{Z \supset V} Z$

$\implies (\text{closure of } V \text{ in } U) = U \cap X = U$

$\text{closure of } V \text{ in } U = \bigcap_{Z \supset V} (Z \cap U)$

$\overset{\text{Lem.}}{\implies} U$ irreducible.

(2) $Y \subset X$. $\quad Y \subset \bar{Y}$, so $Y = \emptyset \iff \bar{Y} = \emptyset$.

$\therefore$ WMA $Y \neq \emptyset$

$Z_1, Z_2 \subset X$ closed

A: $Z_1 \supset Y$	B: $Z_2 \supset Y$	C: $Z_1 \cup Z_2 \supset Y$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
A': $Z_1 \supset \bar{Y}$	B': $Z_2 \supset \bar{Y}$	C': $Z_1 \cup Z_2 \supset \bar{Y}$

by def'n of closure

$$Y \text{ irred.} \Leftrightarrow (C \Rightarrow A \text{ or } B)$$

$$\Updownarrow$$

$$\bar{Y} \text{ irred} \Leftrightarrow (C' \Rightarrow A' \text{ or } B') \qquad \square$$

4. k alg. closed.

(1) $Y \subset \mathbb{A}^n(k)$ closed.

Find a bijection between closed subsets of Y & radical ideals of $k[Y] := \dfrac{k[x_1, \dots, x_n]}{I(Y)}$.

Recall: bijection between radical ideals of $k[\mathbb{A}^n]$ & closed subsets of $\mathbb{A}^n(k)$.

$$\begin{array}{ccc} Z \subset \mathbb{A}^n(k) & \longmapsto & I(Z) \\ & \longleftrightarrow & \\ Z(I) & \longleftarrow\!\shortmid & I \lhd k[x_1, \dots, x_n] \end{array}$$

NB: inclusion-reversing bij.

In our case: $\{\text{closed subsets of } Y\}$

$$\shortparallel$$

$\{\text{closed subsets of } \mathbb{A}^n(k), \text{ contained in } Y\}$

$$\shortparallel$$

$\{\text{radical ideals } I \lhd k[x_1, \dots, x_n] \text{ s.t. } \underbrace{Z(I) \subset Y}_{\Leftrightarrow\, I \supset I(Y)}\}$

I.e. need to prove: $\left\{ \begin{array}{c} \text{radical ideals } I \lhd k[\mathbb{A}^n] \\ I \supset I(Y) \end{array} \right\}$

$$\updownarrow$$

$$\left\{ \text{radical ideals of } k[\mathbb{A}^n]/I(Y) \right\}$$

We know: $A \xrightarrow{p} B$, kernel I

$\rightsquigarrow$ $\{\text{ideals of } A \text{ ctg. } I\} \longleftrightarrow \{\text{ideals of } B\}$

prime $\qquad\qquad$ prime

max $\qquad\qquad$ max

Lemma Under this correspondence, radical ideals are preserved.

I.e.: $J \subset B$. Then J is radial $\Leftrightarrow p^{-1}(J)$ is radial.

Pf J radial $\Leftrightarrow b^n \in J \Rightarrow b \in J$

$\Leftrightarrow p(a)^n \in J \to p(a) \in J$
$\quad$ "$p(a^n)$"

$\Leftrightarrow a^n \in p^{-1} J \Rightarrow a \in p^{-1} J$

$\Leftrightarrow p^{-1} J$ radical. $\square$

(2) If A is a f.t., reduced k-algebra, then $\exists$ alg. set Y over k & iso of k-alg. $k[Y] \cong A$.

f.t.: $k[X_1,\dots,X_n] \overset{p}{\twoheadrightarrow} A$.
kernel I.

reduced $\iff$ I radical: A reduced means (0) is radical
$\Updownarrow$
$p^{-1}(0)$ is radical
$\overset{\shortparallel}{I}$

$\therefore\ A \cong \dfrac{k[X_1,\dots,X_n]}{I}$ ⟵ radical
$\cong k[Z(I)]$ b/c $I(Z(I)) = \sqrt{I}$
$= I$.

$\coprod_{\mathcal{D}} \{*\} \not\hookrightarrow \mathbb{A}^n$ any n

Def^n An algebraic set over k means a closed subset of $\mathbb{A}^n(k)$ with the Zar. top., for some n.

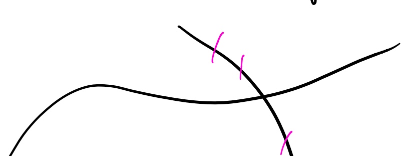
$Z \subset \mathbb{A}^n$

$I = I(Z)$ how many gens does I need?

Fact: $\exists$ finitely many gens

there is a well-defined minimum # of gens.

NB # gens $\underset{\geq}{\neq}$ codimension in general



$f_1, \dots, f_n \in k[\mathbb{A}^r]$

$\rightsquigarrow f : \mathbb{A}^r \longrightarrow \mathbb{A}^n$

$f^{-1}(\{0\}) = Z(f_1, \dots, f_r)$