1. $\alpha : c \to d \in \mathcal{C}$, some cat. TFAE:

(i) α epi

(ii)
$$\begin{tikzcd} c \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & d \arrow[d, equal] \\ d \arrow[r, equal] & d \end{tikzcd}$$
pushout

(iii) $d \amalg_c d$ exists & both maps $d \to d \amalg_c d$ the same.

(i) $\Rightarrow$ (ii)
$$\begin{tikzcd} c \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & d \arrow[d, equal] \arrow[ddr, bend left, "f"] & \\ d & d \arrow[l, equal] \arrow[dr, dashed, "D \ \exists!"] & \\ & & e \arrow[from=2-1, bend right, "g"'] \end{tikzcd}$$

commutativity: $f\alpha = g\alpha$

α epi $\Rightarrow f = g$

$\therefore D = f \ (= g)$ commutes

uniqueness was clear.

(ii) $\Rightarrow$ (iii) $d = d \amalg_c d$ so exists

both maps the identity $\therefore$ equal

(iii) $\Rightarrow$ (i) $f, g : d \to e$ $\quad f\alpha = g\alpha$.

$$\begin{tikzcd} c \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & d \arrow[d, "i_2"] \arrow[dd, bend left, "g"] \\ d \arrow[r, "i_1"] \arrow[dr, bend right, "f"'] & P \arrow[d, "\exists! p"] \\ & e \end{tikzcd}$$

$p i_2 = g$

$p i_1 = f$

but $i_1 = i_2$

$\therefore f = g$

$\therefore \alpha$ epi. $\square$

2. $X \in \mathrm{Top}$

$$G \xleftarrow{\alpha} F \xrightarrow{\beta} H \in \mathrm{PSh}(X).$$

Show that $U \longmapsto G(U) \coprod_{F(U)} H(U)$

defines a pushout in $\mathrm{PSh}(X)$.

$$\begin{tikzcd}
F \arrow[r, "\alpha"] \arrow[d, "\beta"'] & G \arrow[d, "i_2"'] \arrow[dd, bend left, "g"] \\
H \arrow[r, "i_1"] \arrow[dr, bend right, "h"'] & P \arrow[d, dashed, "\exists!\varphi"'] \\
 & T
\end{tikzcd}$$

$P(U) = G(U) \coprod_{F(U)} H(U)$

a presheaf

Why? $U \subset V$

$P(V) \to P(U)$

$i_1(U) : H(U) \to H(U) \coprod_{F(U)} G(U)$

i_2 : similarly

$G(V) \coprod_{F(V)} H(V) \longrightarrow P(U)$

On $G(V) \ni s \longmapsto s|_U$

$H(V) \ni s \longmapsto s|_U$

If $s \in F(V)$ then $\alpha(s)|_U = \alpha(s|_U)$

& $\beta(s)|_U = \beta(s|_U)$

Define p:
$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & G(U) \arrow[d] \arrow[ddr, bend left, "g(U)"] & \\
H(U) \arrow[r] \arrow[drr, bend right, "h(U)"'] & P(U) \arrow[dr, dashed, "p(U)"', "\exists!"] & \circledast \\
& & T(U)
\end{tikzcd}$$

Claim: p is a nat. transf. of presh. i.e. commutes with restriction.

Say $U \subset V$.
$$\begin{tikzcd}
F(V) \arrow[r] \arrow[d] & G(V) \arrow[d] & \\
H(V) \arrow[r] & P(V) \arrow[r, "p(V)"] \arrow[d] & T(V) \arrow[d] \\
& P(U) \arrow[r, "p(U)"'] & T(U)
\end{tikzcd}$$
Commutes?

So STP
$$\begin{tikzcd}
G(V) \arrow[r] \arrow[d] & P(V) \arrow[r] \arrow[d] & T(V) \arrow[d] \\
G(U) \arrow[r] & P(U) \arrow[r] & T(U)
\end{tikzcd}$$
Commutes b/c g nat. of presh.

& Similarly for H.

Uniqueness of p? Yes b/c $p(U)$ is uniquely det'd by commutativity in $\circledast$

Deduce that α epi iff $\alpha(U)$ is epi $\forall U$:

Ex 1: α epi $\iff$ $G \rightrightarrows G \amalg_F G$ both equal

$\iff$ $G(U) \rightrightarrows G(U) \amalg_{F(U)} G(U)$ equal $\forall U$

$\iff$ $\alpha : F(U) \to G(U)$ epi $\forall U$ (Set)

3. $X \in \mathrm{Top}$ $\quad G \leftarrow F \rightarrow H \in \mathrm{Shv}(X)$.

Show: $a(G \amalg^p_F H)$ is pushout.

Deduce: stalks preserve pushouts.

Q: Do sections pres. pushouts?

$\mathcal{C}$ a category: • start with diagram

$$\begin{tikzcd} c \arrow[r] \arrow[d] & d \arrow[d] \\ e \arrow[r] & f \end{tikzcd}$$

Commutes & Universal

example of a colimit "pushout"

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[drr, bend right] \arrow[dr, bend right] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & \cdots \\
 & c \arrow[r, dashed, "\exists!"] & d &
\end{tikzcd}$$

"directed colimit"

$\cdot \leftarrow \cdot \rightarrow \cdot$ — pushout

$\to \to \to \to \cdots$ — directed colimit

$\bullet \quad \bullet$ ~ coproduct

$\cdot \quad - \quad \sim$ ~ other colimits

ex: stalk $F_x = \coprod_{U \ni x} F(U) \Big/ \sim$

$a \in F(U) \sim s \in F(V)$

if $\exists\, W \subset U \cap V$, $W \ni x$, s.t. $a|_W = s|_W$

show: $F_x = \operatorname{colim}_{\substack{\text{open nbhd.} \\ U \ni x}} F(U)$

recall: F presh $\quad H$ sheaf

$$\begin{tikzcd} F \arrow[r] \arrow[d] & H \\ aF =: G \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

this defines the ass. sheaf.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left, "\beta"] & \\ H \arrow[r] \arrow[drr, bend right, "\alpha"'] & P \arrow[r] \arrow[dr, dashed, "\exists!\ \gamma"'] & aP \arrow[d, dashed, "i\exists!\eta"] \\ & & J \end{tikzcd}$$

$F, G, H \in Shv(X)$

P pushout in presh.

J sheaf

- obtain η $\qquad$ unique?
- commutes

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d] \arrow[dd, bend left=60] \\ H \arrow[r] \arrow[dr, bend right] & aP \arrow[d, shift right, "\eta"'] \arrow[d, shift left, "\tilde{\eta}"] \\ & J \end{tikzcd}$$

$i\tilde{\eta} = i\eta$
$= \gamma$

$\therefore \tilde{\eta} = \eta$ by univ. property of γ $\therefore$

$$a : PSh(X) \underset{i}{\overset{a}{\rightleftarrows}} Shv(X) : i$$

"left adjoint" $\qquad$ "right adjoint"

left adjoints always preserve colimits

Deduction: $(aF)_x \cong F_x$

$$\left(G \amalg_F H\right)_x \cong \left[a\left(G \amalg^p_F H\right)\right]_x$$

this is a sheaf

$$\cong \left(G \amalg^p_F H\right)_x$$

$$\cong G_x \amalg_{F_x} H_x$$

$$\operatorname*{colim}_U \left[G(U) \amalg_{F(U)} H(U)\right] = \left(H \amalg^p_F G\right)_x$$

$$\wr\!\wr$$

$$\left[\operatorname*{colim}_U G(U)\right] \amalg_{\operatorname*{colim}_U F(U)} \left[\operatorname*{colim}_U H(U)\right] = G_x \amalg_{F_x} H_x$$

"colimits commute"

4. $\alpha : F \to G \in Shv(X)$. TFAE:

(i) α epi

(ii) $\alpha_x : F_x \to G_x$ epi $\forall x \in X$

(iii) For $U \subset X$ open, $s \in G(U)$ $\exists$ open cover $\{U_i\}$ of U, $s_i \in F(U_i)$ s.t. $\alpha(s_i) = s|_{U_i}$...

⚠: $Sh(X) \hookrightarrow PSh(X)$ need not prs. epi.
(does not)

(i) ⇒ (ii) α epi $\overset{\text{prob. 1}}{\Longleftrightarrow}$ $G \overset{=}{\rightrightarrows} G \amalg_F G$ prob. 3

$\Rightarrow G_x \overset{=}{\rightrightarrows} \left(G_x \amalg_{F_x} G_x\right) = \left(G \amalg_F G\right)_x$

$\underset{\text{prob. 1}}{\Rightarrow}$ α_x is epi

(ii) ⇒ (iii) $U \subset X$ open, $s \in G(U)$. Consider $x \in U$.

α_x surj. $\Rightarrow \exists\, \sigma \in F_x,\ \alpha_x(\sigma) = s_x$.

$\therefore \exists\, \underset{x}{V_x} \subset U$ ($x \in V_x$), $t_x \in F(V_x)$ s.t.

$s|_{V_x} = \alpha(t_x)$.

$\{V_x \mid x \in U\}$ is open cover

∴ done.

(iii) ⇒ (i): $U \subset X$. Shall not show: $F(U) \twoheadrightarrow G(U)$.

Pick $s \in G(U)$. $\exists \{U_i\}$, $s_i \in F(U_i)$ s.t.

$\alpha(s_i) = s|_{U_i}$.

⚠ $\{s_i\}$ need not be compat.

$\alpha(s_i|_{U_i \cap U_j}) = \alpha(s_j|_{U_i \cap U_j})$ but do not learn anything unless α inj.

$$F \xrightarrow{\alpha} G \underset{g}{\overset{f}{\rightrightarrows}} H \qquad f\alpha = g\alpha.$$

Want: $f = g$

Pick $s \in G(U)$. Need: $f(s) = g(s)$.

$\{U_i\}$ cover, $t_i \in F(U_i)$, $\alpha(t_i) = s|_{U_i}$.

$$\begin{array}{ccc} f(\alpha(t_i)) & = & g(\alpha(t_i)) \\ \| & & \| \\ f(s|_{U_i}) & & \\ \| & & \\ f(s)|_{U_i} & & g(s)|_{U_i} \end{array}$$

i.e.: $f(s) = g(s)$ locally

$\therefore f(s) = g(s)$

by unique gluing.