

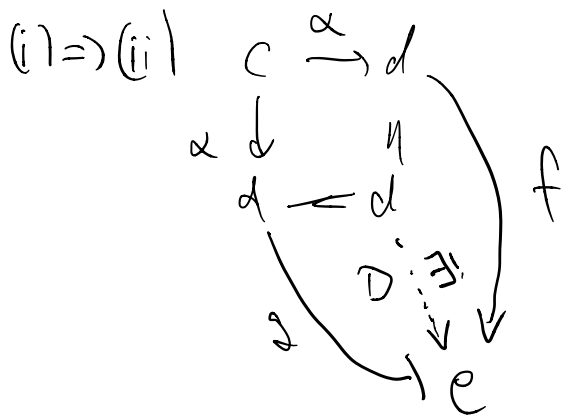
1. $\alpha: c \rightarrow d \in \mathcal{C}$, some cat. TFAE:

(i) α epi

(ii) $c \xrightarrow{\alpha} d$

$\alpha \perp d \parallel$ pushout
 $d = d$

(iii) $\alpha \perp d$ exists & both maps $d \rightarrow d \perp d$ the same.



Commutativity:
 $f\alpha = g$

α epi $\Rightarrow f = g$

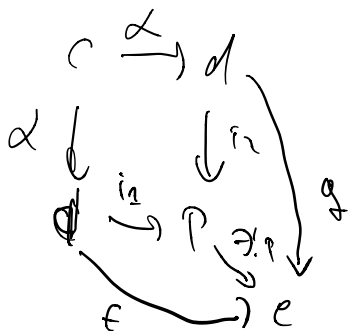
$\therefore D = f (=g)$ commutes

Uniqueness was clear.

(i) \Rightarrow (iii) $\alpha \perp d \perp d$ so exists

Both maps $d \rightarrow d$ the identity \therefore equal

(iii) \Rightarrow (i) $f \circ g: d \rightarrow e$ $f\alpha = g\alpha$.



$\pi_1 = g$

$\pi_2 = f$

But $f_1 = f_2$

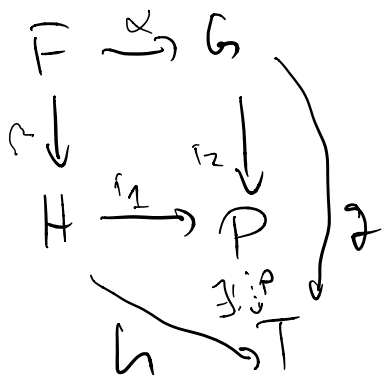
$\therefore f = g$

$\therefore \alpha \text{ epi. } \square$

2. $X \in \text{Top}$

$$\begin{array}{ccc} & F & \\ \alpha \swarrow & & \searrow \beta \\ G & & H \end{array} \in \text{PSL}(X).$$

Show that $U \mapsto \underbrace{G(U) \amalg H(U)}_{F(U)}$ defines a presheaf in $\text{PSL}(X)$.



$$P(U) = \underbrace{G(U) \amalg H(U)}_{F(U)}$$

a presheaf

Why? $U \subset V$

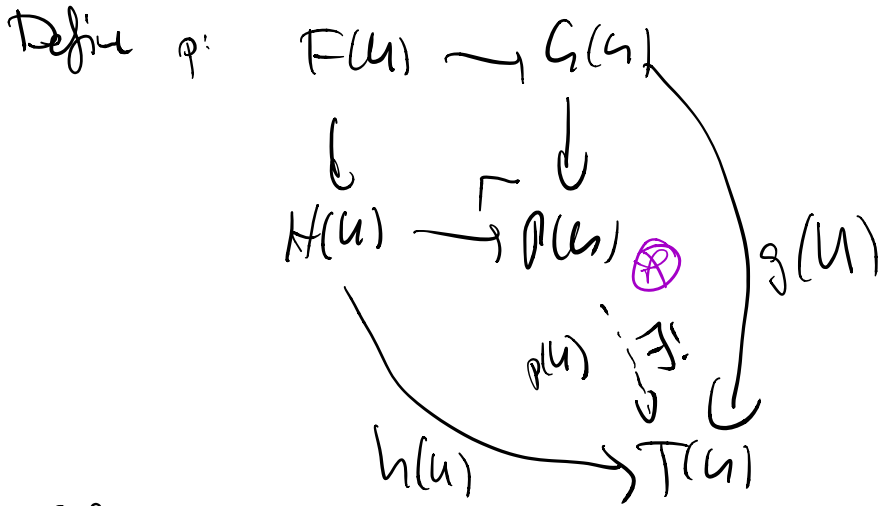
$$P(U) \rightarrow P(V)$$

$$i_2(U): H(U) \rightarrow \underbrace{H(U) \amalg G(U)}_{F(U)} \rightarrow \underbrace{G(U) \amalg H(U)}_{F(V)} \rightarrow P(V)$$

i_1 : similarly

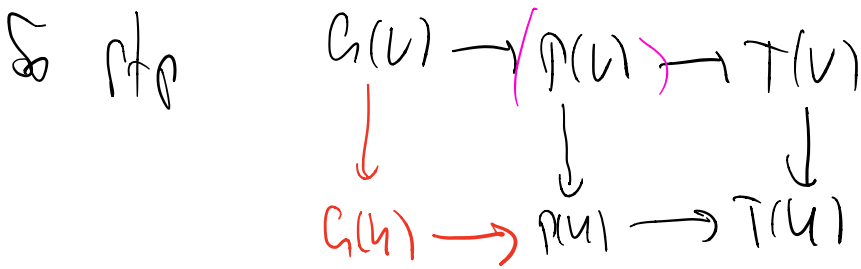
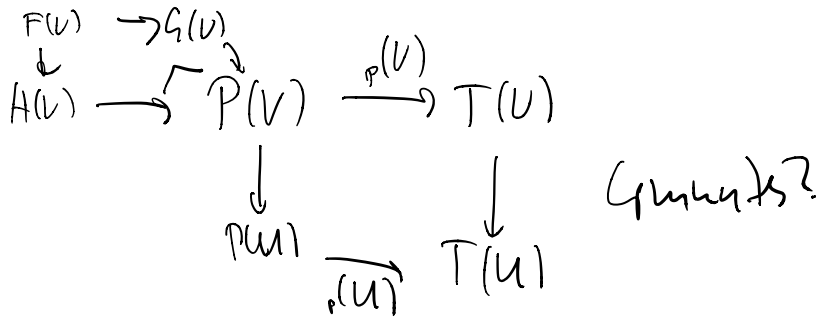
$$\begin{array}{l} G(U) \ni s \mapsto s|_U \\ H(U) \ni s \mapsto s|_U \end{array}$$

If $S \in F(U)$ then $\alpha(s)|_V = \alpha(q)$
& $\beta(s)|_V = \beta(p)$



Claim: ρ is g.w. i.e. commutes with restriction.

ρ prob. say $U \subset V$.



Commutative
g.w. of
prob.

& similarly for H .

Uniqueness of ρ ? Yes bc $\rho(U)$ is uniquely
 det^d by commutativity in \otimes

Deduce that α epi iff $\alpha(U)$ is epi to U :

$$\text{let } \alpha: \alpha \text{ epi} \iff \alpha \twoheadrightarrow \text{Epi} \perp \perp \alpha \text{ both equal}$$

$$\iff \forall U \quad \alpha(U) \twoheadrightarrow \alpha(U) \perp \perp \alpha(U)$$

Epi

$$\iff \alpha: \text{F}(U) \rightarrow \alpha(U) \text{ epi } \forall U$$

(Seq.)

3. $X \in \text{Top}$

$$\begin{array}{ccc} & \text{F} & \\ & \swarrow & \searrow \\ \text{G} & & \text{H} \end{array} \in \text{Shv}(X).$$

Show: $\alpha(\text{G} \perp \perp \text{H})$ is pushout.

Deduce: stalks preserve pushouts.

Q: Do sections preserve pushouts?

\mathcal{C} a category: • start with diagram

$$\begin{array}{ccc} \text{c} & \rightarrow & \text{d} \\ \downarrow & & \downarrow \\ \text{e} & \rightarrow & \text{f} \end{array}$$

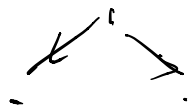
Commutative & Universal

Example of a colimit "pushout"

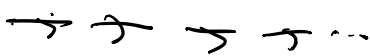
$c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow \dots$



"directed colimit"



— pushout



— directed colimit



— coproduct



— other colimits

Ex: stalk $F_x = \frac{\coprod_{U \ni x} F(U)}{\sim}$

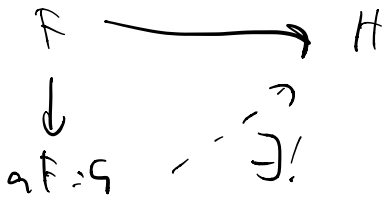
$a \in F(U) \sim s \in F(V)$

if $\exists w \subset U \cap V$

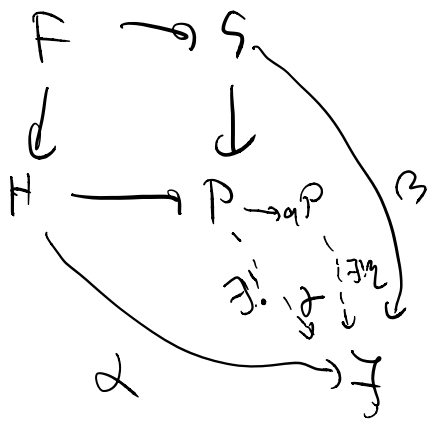
$x \in w$ s.t. $a|_w = s|_w$

Show: $F_x = \text{colim}_{\substack{\text{open } U \ni x \\ U \ni x}} F(U)$

recall: F presh H shunt



this defines the ass. shunt.



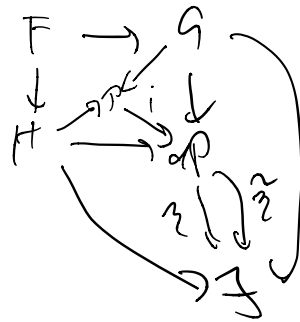
$F, G, H \in \text{Shv}(X)$

P presh and is presh.

\exists shunt

- obtain γ
- commutes

unique?



$$\begin{aligned}
 i\hat{\gamma} &= i\gamma \\
 &= \beta
 \end{aligned}$$

$\therefore \hat{\gamma} = \gamma$ by univ. property of i .

$$a : \text{PSh}(X) \xrightleftharpoons[i]{a} \text{Shv}(X) : i$$

"left adjoint"

"right adjoint"

left adjoints always preserve colimits

Deduction: $(\alpha F)_x \cong F_x$

$$\left(G \underset{F}{\perp} H \right)_x \cong \left[\alpha \left(G \underset{F}{\perp}^p H \right) \right]_x$$

↑
this is a sheaf $\cong \left(G \underset{F}{\perp}^p H \right)_x$

$$\cong G_x \underset{F_x}{\perp} H_x$$

colim $\left[G(U) \underset{F(U)}{\perp} H(U) \right] = \left(G \underset{F}{\perp}^p H \right)_x$

$\left(\underset{G}{\text{colim}} G(U) \right) \underset{\underset{F}{\text{colim}} F(U)}{\perp} \left(\underset{H}{\text{colim}} H(U) \right) = G_x \underset{F_x}{\perp} H_x$

"colimits commute"

4. $\alpha: F \rightarrow G \in \text{Shv}(X)$. **TFAE:**

(i) α epi

(ii) $\alpha_x: F_x \rightarrow G_x$ epi $\forall x \in X$

(iii) for $U \subset X$ open, $s \in G(U) \exists$ open cover $\{U_i\}$ of U , $s_i \in F(U_i)$ s.t. $\alpha(s_i) = s|_{U_i}$

$\mathcal{U}: \text{Sh}(X) \hookrightarrow \text{PSh}(X)$ need not pres. epi.
 (does not)

(i) \Rightarrow (ii) α epi $\stackrel{\text{prob. 1}}{\Leftrightarrow} G \xrightarrow{\cong} G \amalg_F G \stackrel{\text{prob. 3}}{\Rightarrow} G_x \xrightarrow{\cong} (G \amalg_F G)_x \cong (G \amalg_F G)_x$
 $\Rightarrow \alpha_x$ is epi
 \uparrow
 prob. 1

(ii) \Rightarrow (i) $U \subset X$ open, $S \in G(U)$. Consider $x \in U$.
 α_x surj $\Rightarrow \exists \sigma \in F_x, \alpha_x(\sigma) = s_x$.
 $\therefore \exists V_x \subset U, t_x \in F(V_x)$ s.t.
 $s|_{V_x} = \alpha(t_x)$.
 $\{V_x | x \in U\}$ is open cover
 \therefore ~~close~~ ^{not}

(ii) \Rightarrow (i): $U \subset X$. Shall show: $F(U) \twoheadrightarrow G(U)$.
 Pick $S \in G(U)$. $\exists \{U_i\}, s_i \in F(U_i)$ s.t.
 $\alpha(s_i) = s|_{U_i}$.
 $\mathcal{U} \{s_i\}$ need not be compact.
 $\alpha(s|_{\cup U_i}) = \alpha(s|_{\cup U_i})$ Set do not learn

anything unless α
inj.

$$F \xrightarrow{\alpha} G \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} H \quad f\alpha = g\alpha.$$

Want: $f = g$

Pick $s \in G(U)$. Need: $f(s) = g(s)$.

Let $u_i \in U$, $t_i \in F(u_i)$, $\alpha(t_i) = s|_{u_i}$.

$$f(\alpha(t_i)) = g(\alpha(t_i))$$

$$\downarrow$$
$$f(s|_{u_i})$$

$$\downarrow$$
$$f(s)|_{u_i}$$

$$\parallel$$
$$g(s)|_{u_i}$$

i.e.: $f(s) = g(s)$
locally

$\therefore f(s) = g(s)$

by unique gluing.