X top space

- presheaf $\mathcal{F}$ on X means the following data:
  - $\mathcal{F}(U)$ a set, for any $U \subset X$ open
  - $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ a map, $\forall\ V \subset U$ incl. of open sub

Condition: $W \subset V \subset U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[rr, bend right] & \mathcal{F}(V) \arrow[r] \arrow[d, phantom, "\circlearrowleft"] & \mathcal{F}(W) \\ & {} & \end{tikzcd}$$

- a mor. of $\mathcal{F} \to \mathcal{G}$ of presheaves is

$\forall U$: $\mathcal{F}(U) \to \mathcal{G}(U)$

s.t. $\forall\, U \subset V$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{G}(V) \arrow[u] \end{tikzcd}$$

<u>Alternative</u> category Op_X has objects the open subsets of X & morphisms the inclusions.

$$\left[\begin{tikzcd}[row sep=small, column sep=small] U \arrow[r, dashed] \arrow[d] & V \arrow[dl] \\ X & \end{tikzcd}\right]$$

Then cat. of presheaves on X is just

$$\mathrm{Fun}(\mathrm{Op}_X^{op}, \mathrm{Set}).$$

AS "abelian presheaves" "presheaves of ab. grps"

Def^n. A presheaf $\mathcal{F}$ on X is called a <u>sheaf</u> if: $U \subset X$ open

$\{U_i\}$ open covering of U,

then given $\{a_i \in \mathcal{F}(U_i)\}$

s.t. $\forall i j$ $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$

$\exists !\ a \in \mathcal{F}(U)$ s.t. $a|_{U_i} = a_i$.

- The cat. of sheaves on X is the full subcat. of presheaves which happen to be sheaves. I.e. a morphism of sheaves is just a morphism of presheaves.

<u>Ex 1</u> $X \in \mathrm{Top}$, S a set.

$$C(-, S) \in \mathrm{PSh}(X)$$

$$U \longmapsto C(U, S) = \text{cts. maps } U \to S$$

Show: $\mathcal{U}$ is a sheaf.

Given $U \subset X$ open, $\{U_i\}$ open covering

$$f_i : U_i \longrightarrow S \text{ cts}$$

$$\text{s.t. } f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

If $f|_{U_i} = f_i \ \forall i$ then for $x \in X$ $\exists i : x \in U_i$ & $f(x) = f_i(x)$. $\therefore$ at most one gluing

By compat. cond.: indep. of choice of i.

$\therefore f : U \longrightarrow S$ a map.

f is cts. b/c continuity can be checked on open covers.

Def: <u>Associated sheaves.</u> $F \in PSh(X)$.

A morphism $F \longrightarrow G$, where G is a <u>sheaf</u> is called a "sheafification" if given any sheaf H, $F \to H$ any morphism, $\exists!\ G \to H$ s.t.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & H \\ G \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>Comments</u>: G is "unique up to unique iso"

if $\begin{tikzcd} & \arrow[d] \arrow[dr] & \\ & G \arrow[r, dashed, "\exists!"'] & G' \end{tikzcd}$ are both sheafifications & this is iso.

- sheafification always exists, denote it by aF.
- reformulation using more cat. thy:

$$Shv(X) \hookrightarrow PSh(X)$$

admits a left adjoint a.

2. X top. space, S a set.

$$c_S \in PSh(X)$$

$$c_S(U) = S \quad \forall U \quad (\text{rest}^n = id)$$

Show: $a c_S$ = sheaf of locally constant functions valued in S.

$\mathcal{F}(U) = \{$loc. constant functions from U to $S\}$ is a presheaf.

Need to
- find a map $c_S \xrightarrow[i]{} \mathcal{F}$ of presheaves (inclusion of constant functions into loc. const.)
- show that $\mathcal{F}$ is a sheaf
- show that if $\mathcal{G}$ is any sheaf, $c_S \to \mathcal{G}$ any map, then

$$\begin{tikzcd} c_S \arrow[r] \arrow[d] & \mathcal{G} \\ \mathcal{F} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$i : C_S \longrightarrow \mathcal{F}$

over U : $S \longrightarrow \mathcal{F}(U) = \{\text{loc. const. functions } U \to S\}$

$s \longmapsto (x \mapsto s)$

show $\mathcal{F}$ is a sheaf : uniqueness ✓
gluing ✓

$$\begin{tikzcd} C_S \arrow[r, "\varphi"] \arrow[d, "i"'] & \mathcal{G} \\ \mathcal{F} \arrow[ur, dashed, "\tilde{\varphi}"', "\exists!\,?"] & \end{tikzcd}$$

existence : $U \subset X$ open

$\mathcal{F}(U) \ni f : U \to S$ loc. constant

$U = \bigcup_i U_i$, $f_i = f|_{U_i}$ constant

Consider $\varphi(f_i) \in \mathcal{G}(U_i)$.

This is a compat. family: $\varphi(f_i)|_{U_i \cap U_j} = \varphi(f_i|_{U_i \cap U_j})$

$= \varphi(f_j|_{U_i \cap U_j})$

$= \varphi(f_j)|_{U_i \cap U_j}$

$\therefore \exists!\ \tilde{\varphi}(f)$ s.t.

$\tilde{\varphi}(f)|_{U_i} = \varphi(f_i)$.

Check independence of covering:

$U = \bigcup_j V_j \qquad f|_{V_j}$ also constant.

$\rightsquigarrow \quad \tilde{\varphi}(f)' \overset{?}{=} \tilde{\varphi}(f)$

Covering $U_i \cap V_j \rightsquigarrow \tilde{\varphi}(f)\big|_{U_i \cap V_j} = \tilde{\varphi}'(f)\big|_{U_i \cap V_j}$

$\therefore \tilde{\varphi}(f) = \tilde{\varphi}'(f)$ by sheaf axiom.

check: · compatible with restriction

$\rightsquigarrow$ ∴ map of presheaves

· commutes

· unique

$H(U) = \{$ constant functions $U \to S\}$

$H \cong C_S$ $\qquad C_S \xrightarrow{\cong} H$

$$C_S(U) \longrightarrow H(U)$$
$$\overset{\shortparallel}{S} \longmapsto (f_S : x \mapsto s \ \forall x \in U)$$

Defⁿ · $X \xrightarrow{f} Y \in \mathcal{C}$ is a <u>mono</u> if

$\forall \ Z \underset{b}{\overset{a}{\rightrightarrows}} X \qquad fa = fb \Rightarrow a = b$

· epi: $Y \rightrightarrows Z \quad af = bf \Rightarrow a = b$

ex: $\mathcal{C} = \mathrm{Set}$, $Z = \{*\}$ $\quad Z \xrightarrow{a} X \iff a \in X$

$$f \text{ mono} \Rightarrow \left[f(a) = f(b) \Rightarrow a = b \right]$$

(converse also holds)

Similarly f epi $\Leftrightarrow$ f surjective (in Set)

3. (1) Show that $f : A \to B \in \mathrm{Set}$ is epi (i.e. surj) iff the two maps $B \rightrightarrows B \amalg_A B$ are equal.

(2) Show that $f : F \to G \in PSh(X)$ is epi iff $f(U) : F(U) \to G(U)$ is epi of Sets $\forall U$.

Let $\alpha : F \to G$ be a mor. & assume $\alpha(U)$ is epi $\forall U$.

Claim that α is epi.

Let $a, b : G \to H$ s.t. $a\alpha = b\alpha$

i.e. $a(U) = b(U)$ $\forall U \subset X$

but $a(U)\alpha(U) = b(U)\alpha(U)$ & $\alpha(U)$ is epi

$\therefore$ $a(U) = b(U)$ //

<u>Pushouts in Sets</u>

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "i_2"] \\
C \arrow[r, "i_1"] & B \amalg_{{}^f A^g} C
\end{tikzcd} = (B \amalg C)/\sim$$

where $\sim$ is the smallest equiv. rel^n s.t.

$f(a) \sim g(a) \quad \forall a \in A.$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

Show: $f : A \to B$ in Set is epi

$\iff i_1, i_2 : B \to B \coprod_{A} B$ are equal.

- $i_2 f = i_1 f$ by def^n

  $\therefore$ f epi $\Rightarrow i_2 = i_1$

- Conversely: $f(A) \subset B$

$B = f(A) \coprod R$. $\quad f$ epi $\iff R = \emptyset$

$B \coprod_A B = \left[(f(A) \coprod R) \coprod (f(A) \coprod R) \right] / \sim$

$= f(A)/_\sim \coprod R \coprod R$

$B \xrightarrow{i_1, i_2} B \coprod_A B \overset{\sim}{=} f(A)/_\sim \coprod R \coprod R$

$\|$

$f(A) \coprod R$

i_2, i_2

f not.

If $i_1 = i_2$ then $R = \emptyset$ //

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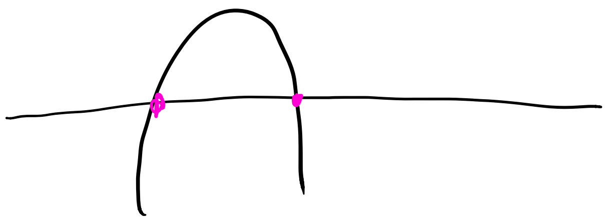
intersection of 2 irred. irreducible?



so so.

- If $f_1, \dots, f_r$ define hypersurfaces then every irred. cpt of $H(f_1) \cap \dots \cap H(f_r)$ inside $\mathbb{A}^n$ has dimension $\geq n-r$.
- If $Z \subset \mathbb{A}^n$ irred, $H \subset \mathbb{A}^n$ hypersurface, $Z \not\subseteq H$, then every cpt. of $Z \cap H$ has dimension $= \dim Z - 1$.

Know: $\dim \geq \dim Z - 1$.

but also: $Z \cap H \subset Z$ proper subset